$SO(3)$-FLOER HOMOLOGY OF 3-MANIFOLDS WITH BOUNDARY 1

KENJI FUKAYA

ABSTRACT. In this paper the author discuss the relation between Lagrangian Floer homology and Gauge-theory (Donaldson theory) Floer homology. It can be regarded as a version of Atiyah-Floer type conjecture in the case of $SO(3)$-bundle with non-trivial second Stiefel-Whitney class. This is a first of a series of papers, where we describe the main results and geometric and algebraic parts of the proof. The half of analytic detail was in [Fu5] which was published in 1998. The other half will appear in subsequent papers.

CONTENTS

1. INTRODUCTION

Let M be an oriented 3 manifold with boundary Σ and $\mathcal{E}_M$ an $SO(3)$ bundle on M such that the restriction of $\mathcal{E}_M$ to each of the connected components of Σ is a nontrivial bundle. (Namely $w_2(\mathcal{E}_M)$ is the fundamental class of Σ.) We denote by $\mathcal{E}_\Sigma$ the restriction of $\mathcal{E}_M$ to Σ. Note Σ is necessary disconnected.

Let $R(M,\mathcal{E}_M)$ (resp. $R(\Sigma,\mathcal{E}_\Sigma)$) be the set of all gauge equivalence classes of flat connections of $\mathcal{E}_M$ on M (resp. on Σ). Restriction of connections define a map $R(M,\mathcal{E}_M) \to R(\Sigma,\mathcal{E}_\Sigma)$. Typically this map is a Lagrangian immersion. More precisely we can perturb $R(M,\mathcal{E}_M)$ so that $R(M,\mathcal{E}_M) \to R(\Sigma,\mathcal{E}_\Sigma)$ becomes a Lagrangian immersion in a way similar to [Fl1, 1(b)], [D1, Section 2(b)]. (See also [He].) Hereafter we write $R(M)$ and $R(\Sigma)$ in place of $R(M,\mathcal{E}_M)$ and $R(\Sigma,\mathcal{E}_\Sigma)$ usually for simplicity.

In [AJ] Akaho-Joyce associated a filtered A_∞ algebra to a (relatively spin) immersed Lagrangian submanifold L of a symplectic manifold X. In this article we

use $\mathbb{Z}_2$ coefficient. Then its underlying module can be taken as

$$CF(L) = H(L;\Lambda_0^{\mathbb{Z}_2}) \oplus \bigoplus_p (\Lambda_0^{\mathbb{Z}_2})^2[p]$$

where the direct sum is taken over all the self-intersection points of L. Here $\Lambda_0^{\mathbb{Z}_2}$ is the universal Novikov ring . (See (2.3) and [FOOO1].)

Our main result is the following.

Theorem 1.1. (1) *The immersed Lagrangian submanifold $R(M)$ is unobstructed in the sense of* [FOOO1], [AJ]. *Namely there exists a bounding cochain b_M of the filtered A_∞-algebra $CF(R(M))$.*

Furthermore there is a canonical choice of b_M. Namely the gauge equivalence class of b_M is an invariant of the pair $(M,\mathcal{E}_M)$.

(2) *Let $(M_1,\mathcal{E}_1)$ and $(M_2,\mathcal{E}_2)$ be pairs of 3 manifolds and bundles with common boundary $(\Sigma,\mathcal{E}) = \partial(M_1,\mathcal{E}_1) = \partial(M_2,\mathcal{E}_2)$, such that $w_2(\mathcal{E}_i|_{\partial M_i}) = [\partial M_i]$, for $i = 1,2$. Let $(M,\mathcal{E})$ be the pair obtained by gluing $(M_1,\mathcal{E}_1)$ and $(-M_2,\mathcal{E}_2)$ along their boundaries. (Here $-M_2$ denote M_2 with orientation reversed.) Then we have a canonical isomorphism*

$$HF(M,\mathcal{E};\Lambda_0^{\mathbb{Z}_2}) \cong HF((R(M_1),b_{M_1}),(R(M_2),b_{M_2})). \tag{1.1}$$

Here the group $HF(M,\mathcal{E};\Lambda_2^{\mathbb{Z}_2})$ is the gauge theory Floer homology of the closed 3 manifold with (nontrivial) $SO(3)$ bundle $\mathcal{E}$. It is defined in [Fl2], [BD]. *See Definition 5.4 for the version with Novikov ring coefficient.*

(3) *If $R(M) \to R(\Sigma)$ is an embedding then $b_M = 0$.*

Statements (2) and (3) imply the next:

Corollary 1.2. *Let $(M,\mathcal{E})$ be a pair of 3 manifold and $SO(3)$ bundle on it which is obtained from $(M_1,\mathcal{E}_1)$ and $-(M_2,\mathcal{E}_2)$ as in Theorem 1.1 (2). We assume $R(M_1)$, $R(M_2)$ are embedded in $R(\Sigma)$. Then:*

$$HF(M,\mathcal{E}) \cong HF(R(M_1),R(M_2)). \tag{1.2}$$

Here the right hand side is the Floer homology of a pair of monotone Lagrangian submanifolds defined by Oh [Oh]. *The isomorphism is one between $\mathbb{Z}_4$ periodic $\mathbb{Z}_2$ vector spaces.*

We may regard Corollary 1.2 as a version of $SO(3)$ analogue of Atiyah-Floer conjecture [At].

Note in case $M_1 = M_2 = \Sigma \times [0,1]$ the isomorphism (1.2) is proved by S. Dostoglou and D.A. Salamon in [DS].

Theorem 1.1 together with other results we will explain below realize the project the author proposed in [Fu1], [Fu2], [Fu4]. We like to mention that there were various proposals such as [LLW], [Sa], [Yo] around the same time (early 1990's).

To put Theorem 1.1 to its natural perspective we first state some other results which are more closely related to the project in [Fu2]. We consider the filtered A_∞ category $\mathscr{FUK}(R(\Sigma))$ whose object is (L,b) where L is an immersed Lagrangian submanifold and b is the bounding cochain of Akaho-Joyce's filtered A_∞ algebra associated to L. In the current situation it is one over $\Lambda_0^{\mathbb{Z}_2}$.

In case we consider only embedded Lagrangian submanifolds as an object such a category $\mathscr{FUK}(R(\Sigma))$ is constructed in [Fu6] and [FOOO3]. Based on the ideas of [AJ] we can enhance it to the version including immersed Lagrangian submanifolds.

Theorem 1.3. *For each $(M, \mathcal{E}_M)$ which bounds $(\Sigma, \mathcal{E}_\Sigma)$ there exists a filtered A_∞ functor $\mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$. Here $\mathscr{CH}$ is the differential graded category (in the sense of* [BK]*) whose object is a chain complex.*

The homotopy equivalence classes of this filtered A_∞ functor is an invariant of $(M, \mathcal{E}_M)$.

We denote this filtered A_∞ functor by $\mathcal{HF}_{(M,\mathcal{E}_M)} : \mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$. The construction of such functor is explained in [Fu1], [Fu2], [Fu4]. (See in particular [Fu4, Section 4].) A part of such idea is realized by Wehrheim [We1] and Salamon-Wehrheim [SaWe] based on a similar but a slightly different analytic setting. We will explain the proof of Theorem 1.3 in Section 2 based on the analytic setting in [Fu5]. (Its detail will appear in [Fu8].)

Theorem 1.4. *The filtered A_∞ functor $\mathcal{HF}_{(M,\mathcal{E}_M)}$ is homotopy equivalent to the filtered A_∞ functor represented by $(R(M), b_M)$.*

Let us elaborate the statement of Theorem 1.4. Let (L, b) be an object of $\mathscr{FUK}(R(\Sigma))$. Theorem 1.3 associates a group

$$HF((M, \mathcal{E}_M), (L, b)) = \mathcal{HF}_{(M,\mathcal{E}_M)}(L, b).$$

More precisely the right hand side is a homology group of the chain complex associated to the object (L, b) by the functor $\mathcal{HF}_{(M,\mathcal{E}_M)}$ in Theorem 1.3. Then Theorem 1.4 claims the existence of an isomorphism

$$HF((M, \mathcal{E}_M), (L, b)) \cong HF((R(M, \mathcal{E}_M), b_M), (L, b)). \tag{1.3}$$

Moreover this isomorphism is functorial in the following sense. Using the fact that $\mathcal{HF}_{(M,E_M)}$ is a filtered A_∞ functor we obtain a map

$$HF((M, \mathcal{E}_M), (L_1, b_1)) \otimes HF((L_1, b_1), (L_2, b_2)) \to HF((M, \mathcal{E}_M), (L_2, b_2)). \tag{1.4}$$

Then the following diagram commutes.

$$\begin{CD}
HF((M, \mathcal{E}_M), (L_1, b_1)) \otimes HF((L_1, b_1), (L_2, b_2)) @>>> HF((M, \mathcal{E}_M), (L_2, b_2)) \\
@VVV @VVV \\
HF((R(M), b_M), (L_1, b_1)) \otimes HF((L_1, b_1), (L_2, b_2)) @>>> HF(R(M, \mathcal{E}_M), (L_2, b_2))
\end{CD}$$

Here the horizontal arrow in the second line is the composition of the morphisms in the filtered A_∞ category $\mathscr{FUK}(R(\Sigma))$. The vertical arrows are induced by (1.3).

We remark that we need to show the commutativity of similar diagrams including higher multiplication operators to obtain an A_∞ functor and prove Theorem 1.4. See [Fu6, Definition 7.1] for the definition of the notion of filtered A_∞ functor. See [Fu6, Definition 7.31] for the definition of the notion of representable filtered A_∞ functor.

The existence of the A_∞ functor in Theorem 1.3 is [Fu4, Theorem 4.8*]. Here * was put to the number of theorems in [Fu4] according to the rule mentioned in [Fu4] page 8 second paragraph, that it, 'Since we postpone the analytic detail to subsequent papers, we put * to the statements which will be proved in subsequent papers.'

It is the understanding of the author that an A_∞ functor of a similar nature in a related context of Heegaard Floer theory of Ozsvath-Szabo is now established, based on more combinatorial method. (See for example [LOT].) The author believes that there is a similar story as those in this paper in the case of Seiberg-Witten

Floer theory, which implies, for example, the coincidence of Seiberg-Witten Floer homology and Heegard Floer homology.

Note Theorems 1.4 and 1.1 (2) together with A_∞ analogue of Yoneda's lemma ([Fu6, Theorem 8.4]. See also [Le].) implies the next corollary.

Corollary 1.5. *Let $(M_i, \mathcal{E}_i)$ be as in Theorem 1.1 (2). Then we have*

$$HF(M, \mathcal{E}_M) \cong H(\mathscr{HOM}(\mathcal{HF}_{(M_1,\mathcal{E}_1)}, \mathcal{HF}_{(M_2,\mathcal{E}_2)})).$$

Here the right hand side is the homology group of the chain complex consisting of all pre-natural transformations from the filtered A_∞ functor $\mathcal{HF}_{(M_1,\mathcal{E}_1)}$ to $\mathcal{HF}_{(M_2,\mathcal{E}_2)}$, which is defined in [Fu6, Definition 7.49] *and* [Fu4, Definition 10.1].

Corollary 1.5 is [Fu1, Conjecture 5.24], [Fu2, Conjecture 3.3] and [Fu6, Conjecture 8.9]. (More precisely it was conjectured there that a particular map defined there gives this isomorphism. We can show the statement in that form also.)[1]

As we mentioned already $M = \Sigma \times [0, 1]$ is an example where $b_M = 0$. In fact $R(M)$ is embedded in $R(\partial M) = R(\Sigma)^2$, as a diagonal. The author has no doubt that there are plenty of examples where $b_M \neq 0$. A possible way to cook up such an example is as follows.

We pretend that the result of this paper holds for $\mathbb{Q}$-coefficient in place of $\mathbb{Z}_2$-coefficient. We take $M_1 = \Sigma \times [0, 1]$. Let M_2 be the connected sum of Poincaré homology sphere and Poincaré homology sphere with orientation reversed. It is proved in [Fu3, Proposition 5.5] that $SU(2)$ Floer homology of M_2 vanishes over $\mathbb{Q}$. So it seems very likely that $\mathcal{HF}_{(M_1\#M_2,\mathcal{E})}$ is represented by the diagonal $R(\Sigma) \subset R(\Sigma)^2$ over $\mathbb{Q}$ coefficient. Here $\mathcal{E}$ is the (unique) $SO(3)$ bundle such that $w_2(\mathcal{E})$ is the generator of $H_2(M; \mathbb{Z}_2) = \mathbb{Z}_2$. On the other hand, using the fact that the fundamental group of the Poincaré homology sphere has 2 irreducible representations over $SO(3)$, the space $R(M)$ has many connected components. (4 connected components diffeomorphic to $SO(3) \times SO(3) \times R(\Sigma)$, 4 connected components diffeomorphic to $SO(3) \times R(\Sigma)$, and one connected components diffeomorphic to $R(\Sigma)$.) Because of $SO(3)$ factors, the space $R(M)$ is not of correct dimension. (Namely its dimension is different from $\dim R(\partial M)/2$.) After perturbation we have still many connected components corresponding to the generators of $H(SO(3)^2)$, $H(SO(3))$ etc.., each of which can be taken to be diffeomorphic to $R(\Sigma)$. The image of them in $R(\partial M)$ all can be taken to be the diagonal. We can perturb so that those components intersect transversally to each other. We thus end up with a complicated configuration of the embedded Lagrangian submanifolds whose connected components are perturbations of the diagonal (and are embedded). The union of such Lagrangian submanifolds is regarded as an immersed Lagrangian submanifold. Together with bounding cochain b_M this immersed Lagrangian submanifold must be $\mathbb{Q}$-coefficient Floer theoretically equivalent to the diagonal. Therefore b_M can not be 0 over $\mathbb{Q}$.

In this example there are ASD connections on $M_2 \times \mathbb{R}$, which gives cancellation over $\mathbb{Q}$ of the generators in the Floer's chain complex of M_2. Those ASD connections are not visible from the space $R(M)$. The bounding cochain b_M however 'remember' those ASD connections.

The author studied the problem we discuss in this paper in 1990's. There are long blank before he restarted it, in this year 2015. During this blank the author

[1] We need to invert the formal generator T of the Novikov ring to prove Corollary 1.5, because A_∞ Yoneda's lemma is proved only after inverting T.

had not been working on this project. In 1990's the present author was close to constructing an A_∞ functor, $\mathcal{HF}_{(M,\mathcal{E}_M)} : \mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$, in a series of papers such as [Fu1], [Fu2], [Fu4], [Fu5]. In this paper we slightly changed the formulation of the construction of this functor. However the changes are mostly due to the development of the Lagrangian Floer theory in last 20 years and are not due to one on the gauge theory part of the story. We apply the language of filtered A_∞ algebra, a module over it, and filtered A_∞ category, which we have developed, especially in [FOOO1], [FOOO2], [Fu6]. We also include immersed Lagrangian submanifolds into the story, based on the work by Akaho-Joyce [AJ]. To include immersed case is essential to formulate Theorem 1.1 (1). Especially it is also crucial for the representativity of the functor $\mathcal{HF}_{(M,\mathcal{E}_M)}$.

Around the time when [Fu5] was written, the author was on the way in establishing and writing up the detailed construction of the moduli spaces which we use in this paper. In fact [Fu5] proved compactness and removable singularity theorem of those moduli spaces. The most important piece of analytic results which was not included in [Fu5] is Fredholm theory, that is, a construction of appropriate Fredholm complex which provides the linearization of the nonlinear equation we study. [2]

The author stopped working on this project for a long time because he could not figure out the way to prove the most important part of the project, which consists of Theorem 1.1 of present paper. Especially he was unable to find a way to prove gluing result, that is, Theorem 1.1 (2). (A gluing result similar to Theorem 1.1 (2) was stated as *Conjecture* 8.9 and not as Theorem 8.9* in [Fu4].) In 1990's the author did not know, either, an appropriate condition under which the relative Floer homology functor $\mathcal{HF}_{(M,\mathcal{E}_M)}$ is determined only by the Lagrangian submanifold $R(M)$. Theorem 1.1 (3) says that the relevant condition is that $R(M) \to R(\Sigma)$ is an *embedding*. Since the author thought that those key issues were yet very hard to attack at that stage he did not work on this project during 1999 - 2014 and concentrated in working out symplectic geometry side of the story.

Meanwhile Wehrheim's paper [We1] appeared. It provides analytic results on a similar moduli space. It is used by Salamon-Wehrheim [SaWe] for a part of the construction of the functor $\mathcal{HF}_{(M,\mathcal{E}_M)} : \mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$.

The argument which we explain in Sections 3,4,5 of present paper resolves the points the author could not go through in 1990's. So we can now prove the results claimed in this section, using the basic properties of the moduli space introduced in [Fu5]. (We explain those moduli spaces in Sections 2,4,5.) I like to mention that the idea used in Sections 4,5 is related to the work by Y. Lekili and M. Lipyanskiy [LL], which they used to study Wehrheim-Woodwards functoriality ([WW1]). (See Remarks 3.12, 4.15, 5.13 for the results we prove on Wehrheim-Woodwards functoriality in a way parallel to the gauge theory results we are discussing here.)

So the point which is the most novel in this paper is an algebraic lemma, Proposition 3.5, and an observation that it can be used to prove Theorem 1.1 (1).

The author is aware that Lipyanskiy have been studying gauge theory Floer homology and Atiyah-Floer conjecture ([Ly]).

Now it's the time to complete the project we started 20 years ago.

---

[2] The reason why the author stopped working on this project during 1998-2014 is *not* because he could not find the way to work out the Fredholm theory.

2. Floer homology of 3 manifolds with boundary as a filtered A_∞ functor: review with update

In this section, we explain the construction of the filtered A_∞ functor, $\mathcal{HF}_{(M,\mathcal{E}_M)}$: $\mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$. The construction is in principle the same as those in [Fu1],[Fu2],[Fu4]. (More precisely it is minor modification of [Fu4, Part 1: geometry]. The other part, [Fu4, Part 2: algebra] was rewritten and was published as [Fu6, Chapter 2].) We however apply the language developed in the past decades systematically for the discussion in this section.

Situation 2.1. M is a 3 manifold with boundary Σ and $\mathcal{E}_M$ is an $SO(3)$-bundle on M such that $\mathcal{E}_\Sigma = \mathcal{E}_M|_\Sigma$ is nontrivial on each of its connected components.

We assume $R(M,\mathcal{E}_M)$, the set of the gauge equivalence classes of flat connections of $\mathcal{E}_M$, is a smooth manifold of dimension $\frac{1}{2}\dim R(\Sigma)$ and the restriction map $i_{R(M)} : R(M,\mathcal{E}_M) \to R(\Sigma,\mathcal{E}_\Sigma)$ is a Lagrangian immersion with transversal self intersection. (More precisely we assume that $H^1(M,d_A) = T_{[A]}R(M,\mathcal{E}_M)$.)

Note we can relax the second half of the condition by perturbing appropriately. In this article we omit the argument to do so for simplicity.

Let L be another immersed Lagrangians submanifold of $R(\Sigma)$ with transversal self-intersection. We write $i_L : \tilde{L} \to R(\Sigma)$ the immersion with image L. We consider the module

$$CF(L) = C(\tilde{L};\Lambda_0^{\mathbb{Z}_2}) \oplus \bigoplus_{(p,q)} \Lambda_0^{\mathbb{Z}_2}[p,q]. \tag{2.1}$$

Here the direct sum runs on the pair of points $(p,q) \in \tilde{L}^2$ such that $p \neq q$ and $i_L(p) = i_L(q)$. In other words we associate two generators to each of the self intersection points, following [Ak], [AJ].

Here and hereafter $C(\tilde{L};\Lambda_0^{\mathbb{Z}_2})$ is a chain model of the (singular) cohomology of $\tilde{L}$. There are various possible choices of the chain model. For example we can take $C(\tilde{L};\Lambda_0^{\mathbb{Z}_2}) = H(\tilde{L};\Lambda_0^{\mathbb{Z}_2})$, the (co)homology group itself. (Here CF stands for Floer chain complex.)

We remark the following isomorphism:

$$CF(L) \cong C(\tilde{L} \times_{R(\Sigma)} \tilde{L};\Lambda_0^{\mathbb{Z}_2}). \tag{2.2}$$

Remark 2.2. We can generalize the story of [AJ] to the case when self-intersection is not necessary transversal but is clean, by taking (2.2) as the definition of $CF(L)$.

We denote by $\Lambda_0^{\mathbb{Z}_2}$ a universal Novikov ring with $\mathbb{Z}_2$ coefficient. (See [FOOO4, (1.3)].) Its element is a formal sum

$$\sum_{i=0}^{\infty} a_i T^{\lambda_0} \tag{2.3}$$

where $0 = \lambda_0 < \lambda_1 < \cdots \uparrow \infty$ and $a_i \in \mathbb{Z}_2$.

Akaho-Joyce (generalizing [FOOO1]) defined a series of operators

$$\mathfrak{m}_k : CF(L)^{k\otimes} \to CF(L)$$

using appropriate moduli spaces of pseudo-holomorphic polygons (See Figure 2.1), and showed that $(CF(L),\{\mathfrak{m}_k\})$ becomes a filtered A_∞ algebra. (See [FOOO1, Definition 3.2.3] for the definition of filtered A_∞ algebra.)

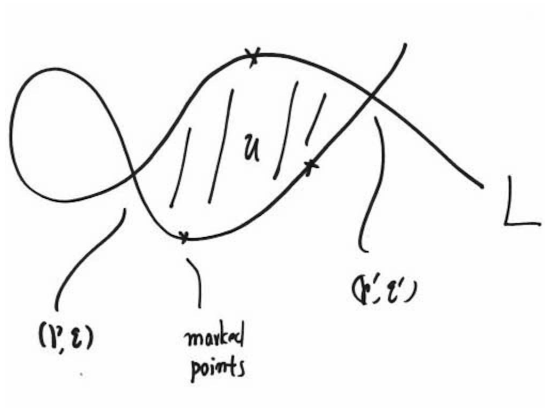

Figure 2.1

Note $\mathfrak{m}_0$ is included. In other words, our filtered A_∞ algebra is curved, in general. We remark that our symplectic manifold $R(\Sigma, \mathcal{E}_\Sigma)$ is monotone. Therefore Lagrangian Floer theory works over $\mathbb{Z}_2$ coefficient by [FOOO5]. Note in [FOOO5] we discussed the case of embedded Lagrangian submanifold. However we can easily combine its argument with [AJ] to include the immersed case. We also remark that in [FOOO5] we assumed that the ambient symplectic manifold is spherically positive but did not require any condition for its Lagrangian submanifold. A montone symplectic manifold is spherically positive.

We denote by $\mathcal{M}(L)$ the set of all the bounding cochains of $(CF(L), \{\mathfrak{m}_k\})$, that is, an element b of $CF(L) \otimes_{\Lambda_0^{\mathbb{Z}_2}} \Lambda_+^{\mathbb{Z}_2}$ such that

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) = 0. \tag{2.4}$$

Here $\Lambda_+^{\mathbb{Z}_2}$ is the maximal ideal of $\Lambda_0^{\mathbb{Z}_2}$ consisting of formal sum (2.3) with $\lambda_0 > 0$.

We put

$$\mathfrak{m}_k^b(x_1, \dots, x_k) = \sum_{\ell_0 \geq 0, \ell_1 \geq 0, \dots, \ell_k \geq 0} \mathfrak{m}_k(b^{\ell_0 \otimes}, x_1, b^{\ell_1 \otimes}, \dots, b^{\ell_{k-1} \otimes}, x_k, b^{\ell_k \otimes}).$$

$(CF(L), \{\mathfrak{m}_k^b\})$ is again a filtered A_∞ algebra and $\mathfrak{m}_0^b = 0$ is equivalent to (2.4).

We refer the reader [FOOO1, Chapter 4] for homological algebra of bounding cochain.

We assume that L is of clean intersection with the Lagrangian immersion $i_{R(M)} : R(M) \to R(\Sigma)$. Namely we assume that

$$\operatorname{Im}(T_x R(M)) \cap \operatorname{Im}(T_y \tilde{L}) \subseteq T_z R(\Sigma)$$

has locally constant rank for $(x, y) \in R(M) \times_{R(\Sigma)} \tilde{L}$ with $z = i_{R(M)}(x) = i_L(y)$ and that this rank coincides with the dimension of the submanifold $R(M) \times_{R(\Sigma)} \tilde{L}$ of $R(M) \times \tilde{L}$, which we assume to be smooth. (Including the case when the intersection is clean but not transversal is important in the discussion of Section 3.)

We consider the fiber product $R(M) \times_{R(\Sigma)} \tilde{L}$, which is a smooth manifold by assumption and put

$$CF((M, \mathcal{E}_M), L) = C(R(M) \times_{R(\Sigma)} L; \Lambda_0^{\mathbb{Z}_2}). \tag{2.5}$$

Here the right hand side is a chain model of ordinary cohomology with $\Lambda_0^{\mathbb{Z}_2}$ coefficient. We define a structure of filtered A_∞ right module on the graded vector space $CF((M,\mathcal{E}_M),L)$ over the filtered A_∞ algebra $(CF(L),\{\mathfrak{m}_k\})$, following and extending the idea of [Fu1], [Fu2], [Fu4]. Note such a structure by definition assigns a series of operators

$$\mathfrak{n}_k : CF((M,\mathcal{E}_M),L)\otimes CF(L)^{k\otimes} \to CF((M,\mathcal{E}_M),L) \tag{2.6}$$

which satisfies the relation

$$\begin{aligned}&\sum_{\ell=0}^{k}\mathfrak{n}_{k-\ell}(\mathfrak{n}_\ell(y;x_1,\dots,x_\ell);x_{\ell+1},\dots,x_k)\\ &+\sum_{0\le\ell\le m\le k}\mathfrak{n}_{k-m+\ell+1}(y;x_1,\dots,\mathfrak{m}_{m-\ell}(x_\ell,\dots,x_{m-1}),\dots,x_k)=0.\end{aligned} \tag{2.7}$$

(See [FOOO1, Section 3.7.1]. Note there it was discussed the case of C_1-C_2 filtered A_∞ bimodule D. The case of right filtered A_∞ module is its special case where C_1 is Λ_0.). For $[b]\in\mathcal{M}(L)$ we define $d^b : CF((M,\mathcal{E}_M),L)\to CF((M,\mathcal{E}_M),L)$ by

$$d^b(y)=\sum_{k=0}^{\infty}\mathfrak{n}_k(y;b,\dots,b). \tag{2.8}$$

(2.4) and (2.7) imply that

$$d^b\circ d^b=0. \tag{2.9}$$

(See [FOOO1, Lemma 3.7.14].) We now define a $\Lambda_0^{\mathbb{Z}_2}$ module

$$HF((M,\mathcal{E}_M),(L,b))=\frac{\operatorname{Ker}d^b}{\operatorname{Im}d^b}. \tag{2.10}$$

In the rest of this section we explain the construction of the maps $\mathfrak{n}_k$.

We take a Riemannian metric on M such that there exists a compact subset M_0 and an isometry

$$M\setminus M_0\cong(-1,1]\times\Sigma,$$

where $\partial M=\Sigma$ corresponds to $\{1\}\times\Sigma$. Here the metric of the right hand side is the direct product metric with a Kähler metric g_Σ on Σ. We fix g_Σ hereafter. Note g_Σ determines a complex structure of $R(\Sigma)$. We take a smooth function $\chi:(-1,1]\to[0,1]$ such that $\chi(s)\equiv 1$ on a neighborhood of $\{-1\}$ and

$$\{s\mid\chi(s)=0\}=[0,1].$$

We also assume

$$\chi(s)=e^{1/s}\qquad\text{for } s\in(-\epsilon,0).$$

We consider the 'ASD-equation' of 4 manifolds $M\times\mathbb{R}$ with respect to the singular metric

$$\mathbf{g}=\begin{cases}g_M+dt^2 & \text{on } M_0\times\mathbb{R}\\ \chi(s)^2g_\Sigma+ds^2+dt^2 & \text{on } (M\setminus M_0)\times\mathbb{R}.\end{cases} \tag{2.11}$$

Note we identify

$$M\setminus M_0\cong\Sigma\times(-1,1]\times\mathbb{R}$$

and use s (resp. t) for the coordinate of the $(-1,1]$ (resp. $\mathbb{R}$) factor.

Since the metric $\mathbf{g}$ is singular the ADS equation

$$F_{\mathfrak{A}}+*_{\mathbf{g}}F_{\mathfrak{A}}=0 \tag{2.12}$$

does not make sense. Here $\mathfrak{A}$ is a connection of the $SO(3)$ bundle $\mathcal{E}_M \times \mathbb{R}$, $F_{\mathfrak{A}}$ its curvature, and $*\mathbf{g}$ is the Hodge $*$ operator of the 'metric' $\mathbf{g}$. However we can make sense of the equation (2.12) as follows. We write the restriction of $\mathfrak{A}$ to $M \setminus M_0 \cong \Sigma \times (-1,1] \times \mathbb{R}$ as

$$\mathfrak{A} = A + \Phi ds + \Psi dt \tag{2.13}$$

where $A = A(s,t)$ is a $(s,t) \in (-1,1] \times \mathbb{R}$ parametrized family of connections of $\mathcal{E}_\Sigma$ and Φ, Ψ are $(s,t) \in (-1,1] \times \mathbb{R}$ parametrized families of the sections of $so(3)$-bundle associated to $\mathcal{E}_\Sigma$ by the adjoint representation of $SO(3)$ on $so(3)$. Then on the domain $\Sigma \times (-1,0) \times \mathbb{R}$ where $\mathbf{g}$ is indeed a Riemannian metric and (2.12) makes sense, we can rewrite Equation (2.12) as follows.

$$\begin{aligned} &\frac{\partial A}{\partial t} - d_A\Psi - *_\Sigma\left(\frac{\partial A}{\partial s} - d_A\Phi\right) = 0, \\ &\chi(s)^2\left(\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi,\Psi]\right) + *_\Sigma F_A = 0. \end{aligned} \tag{2.14}$$

See [DS], [Fu5]. We observe that Equation (2.14) makes sense also when $\chi(s) = 0$. In that case we may regard the solution of (2.14) as a holomorphic map $[0,1] \times \mathbb{R} \to R(\Sigma)$ as follows. When $\chi(s) = 0$, the second equation means that F_A, the curvature of the connection A (of $\mathcal{E}_\Sigma$), is 0. Namely $A(s,t)$ is a flat condition. Therefore $(s,t) \mapsto [A(s,t)]$ defines a map $[0,1] \times \mathbb{R} \to R(\Sigma)$. The first equation implies that this map is holomorphic as follows. Note $\frac{\partial A}{\partial s} - d_A\Phi$ and $\frac{\partial A}{\partial t} - d_A\Psi$ are d_A-closed forms since $A(s,t)$ is flat for $(s,t) \in (0,1] \times \mathbb{R}$. Therefore the first equation implies that $\frac{\partial A}{\partial s} - d_A\Phi$ and $\frac{\partial A}{\partial t} - d_A\Psi$ are both harmonic. (Namely they are both d_A and d_A^* closed). They represent the s (resp. t) derivative of our map $(s,t) \mapsto [A(s,t)]$. The tangent space $T_{A(s,t)}R(\Sigma)$ is identified with the set of harmonic forms. The complex structure of $R(\Sigma)$ is obtained by the Hodge $*_\Sigma$ on harmonic forms. (See, for example, [Fu5, Section 2].) Thus the first equation implies that $(s,t) \mapsto [A(s,t)]$ is holomorphic.

The operator (2.6) is obtained by 'counting' the order of certain moduli space of solutions of (2.12), (2.14) with appropriate boundary conditions, which we will describe below. We first discuss the case when L is an embedded Lagrangian submanifold and will explain its generalization to the immersed case later.

Let $\mathfrak{A}$ be a solution of (2.12),(2.14). (Namely we require (2.12) on $M_0 \times \mathbb{R}$ and (2.14) on $(M \setminus M_0) \times \mathbb{R}$.) We define the energy $\mathcal{E}(\mathfrak{A})$ as follows.

$$\mathcal{E}(\mathfrak{A}) = \int_{(M\setminus(\Sigma\times[0,1)))\times\mathbb{R}} \|F_{\mathfrak{A}}\|^2 \Omega_{\mathbf{g}} + \int_{[0,1)\times\mathbb{R}} \left(\left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2\right) dsdt. \tag{2.15}$$

Here $\Omega_{\mathbf{g}}$ is the volume form of the metric $\mathbf{g}$ and we define u by $u(s,t) = [A(s,t)] \in R(\Sigma)$. The norm appearing in the second term of the right hand side is the norm of the vector. The norm is induced by the metric obtained by using g_Σ. See [Fu5, Section 2].

Hereafter we write

$$\|\nabla u\|^2 = \left\|\frac{\partial u}{\partial s}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2.$$

We assume that L is of clean intersection with $R(M)$. Let R_1 and R_2 be connected components of the fiber product $R(M) \times_{R(\Sigma)} \tilde{L}$. We consider the connection

$\mathfrak{A}$ solving (2.12), (2.14). As we explained above $(s,t) \mapsto [A(s,t)]$ defines a holomorphic map $[0,1] \times \mathbb{R} \to R(\Sigma)$. We consider the following boundary condition:[3]

Condition 2.3. There exists a smooth map $\gamma : (-\infty, +\infty) \to L$ such that

$$[A(1,t)] = i_L(\gamma(t)) \in R(\Sigma). \tag{2.16}$$

Here $i_L : L \to R(\Sigma)$ is the Lagrangian embedding .

See Figure 2.2.

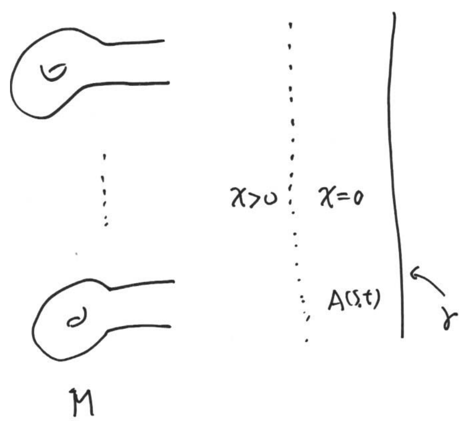

Figure 2.2

We have the following:

Theorem 2.4. *Suppose* $\mathfrak{A}$ *is a solution of (2.12), (2.14). We assume that its energy is finite. Then there exists a gauge transformation* $g \in \mathrm{Aut}(\mathcal{E}_M \times \mathbb{R})$ *and* $a_-, a_+ \in R(M)$ *with the following properties.*

1. $\lim_{t\to\infty} g^*\mathfrak{A}|_{M\times\{t\}} = a_+$, *and* $\lim_{t\to-\infty} g^*\mathfrak{A}|_{M\times\{t\}} = a_-$.
2. *If* $A(s,t)$ *is obtained by the restriction of* $\mathfrak{A}$ *to* $\Sigma \times \{(s,t)\}$ *then* $[A(1,t)] = \gamma(t)$.
3. $\lim_{t\to-\infty} i_L(\gamma(t)) = [a_-|_\Sigma]$, $\lim_{t\to+\infty} i_L(\gamma(t)) = [a_+|_\Sigma]$.

Here $[*]$ stands for the gauge equivalence class of the connection $*$.

We did not specify the ratio of the convergence in Theorem 2.4 (1). Actually we can prove that there exist positive numbers c_k, C_k such that:

$$\|g^*\mathfrak{A}|_{M\times\{t\}} - a_\pm\|_{C^k} \le C_k e^{\mp c_k t}. \tag{2.17}$$

We remark that (2.17) implies that the convergence in item (3) is also of exponential order.

Remark 2.5. We postpone the proof of Theorem 2.4 to [Fu7], [Fu8].

In the case when M has no boundary and $R(M)$ is zero dimensional, Floer [Fl1] proved Theorem 2.4 together with the estimate (2.17). In the case M has no boundary and $R(M)$ is of Bott-Morse type, that is, the critical points set $R(M)$ of the Chern-Simons functional is of Bott-Morse type, Theorem 2.4 together with the estimate (2.17) is proved in [Fu3, Lemma 7.13]. See also [MMR], where (in the

[3] In case L is immersed we need to set a boundary condition a bit more carefully. See Definition 2.23 (6), (7).

case M has no boundary) Theorem 2.4 is proved *without* assuming that $R(M)$ is clean in the Bott-Morse sense. (In that generality the estimate (2.17) is false.) We need certain modification of the proof of [Fu3] to prove Theorem 2.4 and estimate (2.17).

A similar result is proved in [SaWe, Theorem 5.1] in a slightly different setting.

We now define:

Definition 2.6. We define the moduli space $\overset{\circ}{\widetilde{\mathcal{M}}}((M,\mathcal{E}),L;R_-,R_+;E)$ as the set of all the gauge equivalence classes of $\mathfrak{A}$ such that:

(1) $\mathfrak{A}$ satisfies (2.12), (2.14).
(2) There exists $\gamma : (-\infty,+\infty) \to \tilde{L}$ such that Condition 2.3 is satisfied.
(3) Let a_- and a_+ be as in the conclusion of Theorem 2.4. Then
$$a_- \in R_-, \qquad a_+ \in R_+. \tag{2.18}$$
(4) $\mathcal{E}(\mathfrak{A}) = E$, where the left hand side is the energy defined by (2.15).

$\overset{\circ\circ}{\widetilde{\mathcal{M}}}((M,\mathcal{E}),L;R_-,R_+;E)$ has a natural $\mathbb{R}$ action defined by the translation on $\mathbb{R}$ direction. We denote by $\overset{\circ\circ}{\mathcal{M}}((M,\mathcal{E}),L;R_-,R_+;E)$ the quotient of $\overset{\circ\circ}{\widetilde{\mathcal{M}}}((M,\mathcal{E}),L;R_-,R_+;E)$ by this $\mathbb{R}$ action.

In the same way as [Fl2, 1(b)], [D1, Section 2 (b)], we can perturb our equation (2.12), (2.14) on $M_0\times\mathbb{R}$ (that is, at the gauge theory part) so that $\overset{\circ\circ}{\mathcal{M}}((M,\mathcal{E}),L;R_-,R_+;E)$ becomes a smooth manifold. Hereafter in this paper we denote by $\overset{\circ\circ}{\mathcal{M}}((M,\mathcal{E}),L;R_-,R_+;E)$ this perturbed moduli space.

We next describe a partial compactification of $\overset{\circ\circ}{\mathcal{M}}((M,\mathcal{E}),L;R_-,R_+;E)$. Our compactification is a mixture of Uhlenbeck compactification in the gauge theory side and of stable map compactification in the (pseudo)holomorphic curve side.

Let Ω be a bordered nodal curve such that:

Condition 2.7. (1) Ω contains $(0,1]\times\mathbb{R}$ as its irreducible component.
(2) $\overline{\Omega \setminus ((0,1]\times\mathbb{R})}$ is a finite union of trees of sphere components attached to $(0,1)\times\mathbb{R}$ and a finite union of trees of disk components attached to $\{1\}\times\mathbb{R}$.

For each tree of disk or sphere components, its root is by definition its intersection with $(0,1]\times\mathbb{R}$. Note disk components in Condition 2.7 (2) may contain a tree of sphere components attached to it.

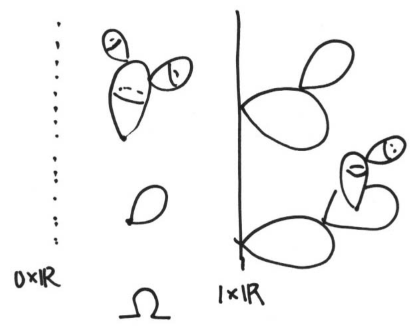

Figure 2.3

We consider a pair (Ω, u) where Ω satisfies Condition 2.7 and u is a map : $\Omega \to R(\Sigma)$ which satisfies the next condition.

Condition 2.8. (1) There exists a continuous map $\gamma : \partial\Omega \setminus \{\text{boundary nodes}\} \to L$ such that $u(z) = (i_L \circ \gamma)(z)$.
(2) u is holomorphic on each of the irreducible components of Ω and is continuous on Ω.
(3) (Ω, u) is stable. Namely the set of all maps $v : \Omega \to \Omega$ satisfying the next three conditions is a finite set.
   (a) v is a homeomorphism and is holomorphic on each of the irreducible components.
   (b) v is the identity map on $(0, 1] \times \mathbb{R} \subseteq \Omega$.
   (c) $u \circ v = u$.

Definition 2.9. We define the set $\overset{\circ}{\widetilde{\mathcal{M}}}((M, \mathcal{E}_M), L; R_-, R_+; E)$ as the set of all equivalence classes of $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ satisfying the following conditions.

(1) $\mathfrak{A}$ is a connection of $\mathcal{E}_M \times \mathbb{R}$ satisfying equations (2.12), (2.14).
(2) $\mathfrak{z} = (\mathfrak{z}_1, \dots, \mathfrak{z}_{m_1})$ is an *unordered* m_1-tuple of points of $M \setminus (\Sigma \times [0, 1]) \times \mathbb{R}$. We put $\|\mathfrak{z}\| = m_1$. We say the subset $\{\mathfrak{z}_1, \dots, \mathfrak{z}_{m_1}\} \subset M \setminus (\Sigma \times [0, 1]) \times \mathbb{R}$ the *support* of $\mathfrak{z}$ and denote it by $|\mathfrak{z}|$. We define multi : $|\mathfrak{z}| \to \mathbb{Z}_{>0}$ by $\text{multi}(x) = \#\{i \mid z_i = x\}$ and call it the *multiplicity function.*
(3) $\mathfrak{w} = (\mathfrak{w}_1, \dots, \mathfrak{w}_{m_2})$ is an *unordered* m_2-tuple of points of $\{1\} \times \mathbb{R}$. We put $\|\mathfrak{w}\| = m_2$. We say the subset $\{\mathfrak{w}_1, \dots, \mathfrak{w}_{m_2}\} \subset \{1\} \times \mathbb{R}$ the *support* of $\mathfrak{w}$. We define multi : $|\mathfrak{w}| \to \mathbb{Z}_{>0}$ by $\text{multi}(x) = \#\{i \mid w_i = x\}$ and call it the *multiplicity function.*
(4) Ω satisfies Condition 2.7.
(5) (Ω, u) satisfies Condition 2.8.
(6) For $(s, t) \in (0, 1] \times \mathbb{R} \subseteq \Omega$ we have
$$[A(s, t)] = u(s, t).$$
Here $A(s, t)$ is obtained from $\mathfrak{A}$ by (2.13).
(7) The energy of $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ which we will define in Definition 2.10 below is E.
(8) Definition 2.6 (3) holds.

The equivalence relation is defined in Definition 2.11 below.

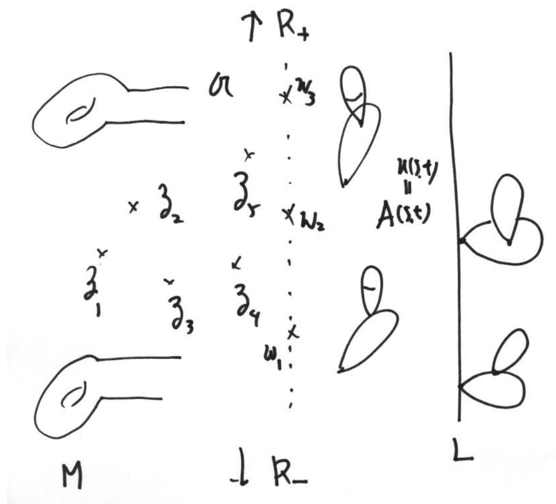

Figure 2.4

Definition 2.10. Suppose $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ satisfies (1)-(6) of Definition 2.9. We define its *energy* $\mathcal{E}(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ by the next formula:

$$\begin{aligned} \mathcal{E}(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u) = &\int_{(M\setminus(\Sigma\times[0,1)))\times\mathbb{R}} \|F_{\mathfrak{A}}\|^2 \Omega_{\mathbf{g}} \\ &+ \int_{\Sigma} \|\nabla u\|^2 ds dt + 2\pi^2\|\mathfrak{z}\| + 2\pi^2\|\mathfrak{w}\|. \end{aligned} \tag{2.19}$$

Definition 2.11. We say $(\mathfrak{A}_1, \mathfrak{z}_1, \mathfrak{w}_2, \Omega_1, u_1)$ is *equivalent* to $(\mathfrak{A}_2, \mathfrak{z}_2, \mathfrak{w}_2, \Omega_2, u_2)$ if the following holds.

(1) There exists a gauge transformation g such that $g^*\mathfrak{A}_1 = \mathfrak{A}_2$.
(2) $\mathfrak{z}_1 = \mathfrak{z}_2$. $\mathfrak{w}_1 = \mathfrak{w}_2$.
(3) There exists a map $v : \Omega_1 \to \Omega_2$ such that:
  (a) v is a homeomorphism and is holomorphic on each of the irreducible components.
  (b) v is the identity map on $(0, 1] \times \mathbb{R} \subseteq \Omega$.
  (c) $u_2 \circ v = u_1$.

We define a topology on $\overset{\circ}{\widetilde{\mathcal{M}}}((M, \mathcal{E}), L; R_-, R_+; E)$ by combining the topology of Uhlenbeck compactification and stable map topology as follows.

Definition 2.12. Let $[\mathfrak{A}_n, \mathfrak{z}_n, \mathfrak{w}_n, \Omega_n, u_n]$ be a sequence of elements of the set $\overset{\circ}{\widetilde{\mathcal{M}}}((M, \mathcal{E}), L; R_-, R_+; E)$ and $[\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u] \in \overset{\circ}{\widetilde{\mathcal{M}}}((M, \mathcal{E}), L; R_-, R_+; E)$. We say $[\mathfrak{A}_n, \mathfrak{z}_n, \mathfrak{w}_n, \Omega_n, u_n]$ converges to $[\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u]$ if the following holds.

(1) Let $|\mathfrak{z}| \subset (M \times \mathbb{R}) \setminus (\Sigma \times [0, 1] \times \mathbb{R})$ be the support of $\mathfrak{z}$. We require that there exists a sequence of gauge transformations g_n such that $g_n^*\mathfrak{A}_n$ converges to $\mathfrak{A}$ in compact C^∞ topology on $(M \times \mathbb{R}) \setminus (\Sigma \times [0, 1] \times \mathbb{R}) \setminus |\mathfrak{z}|$.
(2) For $\epsilon > 0$ we denote by $\Omega_n(\epsilon) \subset \Omega_n$ the domain $\Sigma \times [\epsilon, 1] \times \mathbb{R}$ together with all the trees of sphere and disc components of Ω_n whose roots are in $\Sigma \times [\epsilon, 1] \times \mathbb{R}$. We define $\Omega(\epsilon) \subset \Omega$ in the same way.

Then, for any ϵ such that the root of sphere components of (Ω, u) are not on $\{\epsilon\} \times \mathbb{R}$, we require $(\Omega_n(\epsilon), u_n)$ converges to $(\Omega(\epsilon), u)$ in stable map topology, which is defined in the same way as [FOn, Definition 10.3].

(3) Let $x \in |\mathfrak{z}|$. Then for each sufficiently small $\epsilon > 0$ the next equality holds. Here multi is the multiplicity function and $B_\epsilon(x)$ is the metric ball centered at x in $M \times \mathbb{R}$.

$$\begin{aligned} &2\pi^2 \text{multi}(x) + \int_{B_\epsilon(x)} \|F_{\mathfrak{A}}\|^2 \Omega_{\mathbf{g}} \\ &= \lim_{n\to\infty} \left(\sum_{y \in B_\epsilon(x) \cap |\mathfrak{z}_n|} 2\pi^2 \text{multi}(y) + \int_{B_\epsilon(x)} \|F_{\mathfrak{A}_n}\|^2 \Omega_{\mathbf{g}} \right). \end{aligned} \tag{2.20}$$

(4) Let $x \in |\mathfrak{w}| \subset \{0\} \times \mathbb{R}$. We define $D_\epsilon(x, i)$ $i = 1, 2, 3$ and $D_\epsilon(x, 4; \Omega)$, $D_\epsilon(x, 4; \Omega_n)$ as follows.
   (a) $D_\epsilon(x, 1) = \Sigma \times (D_\epsilon(x) \cap ([-1, 0) \times \mathbb{R}))$. Here $D_\epsilon(x)$ is the metric ball centered at x in $[-1, 1] \times \mathbb{R}$.
   (b) $D_\epsilon(x, 2) = (\{0\} \times \mathbb{R}) \cap D_\epsilon(x)$.
   (c) $D_\epsilon(x, 3) = ((0, 1] \times \mathbb{R}) \cap D_\epsilon(x)$.
   (d) $D_\epsilon(x, 4; \Omega)$ is a subset of Ω and is the union of $D_\epsilon(x, 3)$ and the trees of sphere components are rooted on $D_\epsilon(x, 3)$. The definition of $D_\epsilon(x, 4; \Omega_n)$ is similar.

We then require the next equality for sufficiently small positive numbers ϵ.

$$\begin{aligned} &2\pi^2 \text{multi}(x) + \int_{D_\epsilon(x;1)} \|F_{\mathfrak{A}}\|^2 \Omega_{\mathbf{g}} + 2 \int_{D_\epsilon(x,4;\Omega)} \|\nabla u\|^2 ds dt \\ &= \lim_{n\to\infty} \left(\sum_{y \in D_\epsilon(x,1) \cap |\mathfrak{z}_n|} 2\pi^2 \text{multi}(y) + \int_{D_\epsilon(x,1)} \|F_{\mathfrak{A}_n}\|^2 \Omega_{\mathbf{g}} \right. \\ &\qquad \left. + \int_{D_\epsilon(x,4;\Omega_n)} \|\nabla u\|^2 ds dt + \sum_{y \in D_\epsilon(x,2) \cap |\mathfrak{w}_n|} 2\pi^2 \text{multi}(y) \right). \end{aligned} \tag{2.21}$$

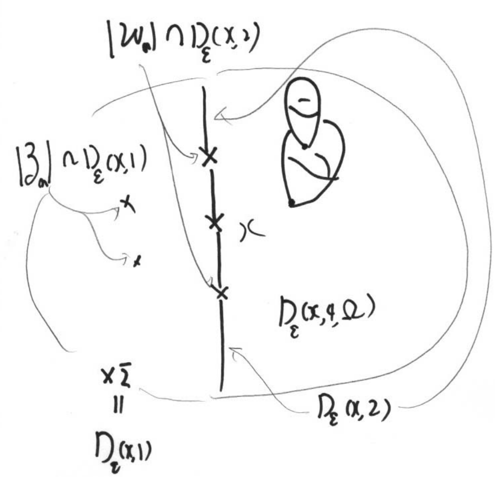

Figure 2.5

Remark 2.13. This topology is a combination of the topology of Uhlenbeck compactification defined in [D2, page 292] and the stable map topology introduced in [FOn, Definition 10.2].

The unordered finite set $\mathfrak{z}$ plays the role to record the total mass of the bubble in the gauge theory side. It appeared in Uhlenbeck compactification in gauge theory.

The unordered set $\mathfrak{w}$ plays the role to record the total mass of the bubble which occurs on the line $\{0\} \times \mathbb{R}$. The bubble on this line is studied in detail in [Fu5]. Note this is a subset of $\{0\} \times \mathbb{R}$ and is not a subset of $\Sigma \times \{0\} \times \mathbb{R} \subset M \times \mathbb{R}$. Actually we can not specify where bubble occurs on $\Sigma \times \{0\} \times \mathbb{R} \subset M \times \mathbb{R}$ by the following reason. Let $x \in \{0\} \times \mathbb{R}$. The sequence $\mathfrak{A}_n$ determines a sequence of maps $u_n : B_x(\epsilon) \setminus \text{a finite set} \to R(\Sigma)$ which is holomorphic. (See [Fu5, Lemmata 5.5 and 4.22].) Even in case u_n is defined at x the limit of u_n may not be defined at x. Namely the bubble in the symplectic geometry side may occur on the line $\{0\} \times \mathbb{R}$. In such a case the connection $\mathfrak{A}_n$ may diverge everywhere on $\Sigma \times \{0\} \times \{t\}$.

Item (4) also takes into account the following phenomenon. There may be a sequence of trees of sphere components of Ω_n whose roots are (s_n, t_n) where $s_n > 0$ converges to 0 and t_n converges to t. Then in the limit this sequence of sphere components moves to $(0, t)$, which lies on the line $\{0\} \times \mathbb{R}$. Therefore the limit is no longer contained in Ω. In this case, we take $(0, t)$ as a part of $\mathfrak{w}$ and its multiplicity is the limit of the symplectic area of those trees of the sphere components.

It may also happen that some of the points of $\mathfrak{z}_n$ converges to $\Sigma \times \{0\} \times \mathbb{R}$. In that case it will become one of the points of $\mathfrak{w}$, forgetting the Σ factor.

Definition 2.14. $\widetilde{\overset{\circ}{\mathcal{M}}}((M, \mathcal{E}), L; R_-, R_+; E)$ has an $\mathbb{R}$ action by translation along the $\mathbb{R}$ factor. We denote by $\overset{\circ}{\mathcal{M}}((M, \mathcal{E}), L; R_-, R_+; E)$ the quotient space with quotient topology.

We define a continuous map

$$\mathrm{ev}_\pm : \overset{\circ}{\mathcal{M}}((M, \mathcal{E}), L; R_-, R_+; E) \to R_\pm \tag{2.22}$$

by using Definition 2.9 (8).

We define $\mathcal{M}((M, \mathcal{E}), L; R_-, R_+; E)$ as the disjoint union of fiber products:

$$\begin{aligned} &\overset{\circ}{\mathcal{M}}((M, \mathcal{E}), L; R_-, R_1; E_0) \times_{R_1} \overset{\circ}{\mathcal{M}}((M, \mathcal{E}), L; R_1, R_2; E_1) \times_{R_2} \dots \\ &\dots \times_{R_{\ell-1}} \overset{\circ}{\mathcal{M}}((M, \mathcal{E}), L; R_{\ell-1}, R_\ell; E) \times_{R_\ell} \mathcal{M}((M, \mathcal{E}), L; R_\ell, R_+; E_\ell) \end{aligned} \tag{2.23}$$

where

$$E = E_0 + E_1 + \cdots + E_\ell, \tag{2.24}$$

$R_1, \dots, R_\ell$ are connected components of $R(M) \times_{R_\Sigma} L$ and $\ell = 0, 1, 2, \dots$. Using Theorem 2.4, we can define a topology on $\mathcal{M}((M, \mathcal{E}), L; R_-, R_+; E)$ in the same way as [FOOO1, Section 7.1.4].

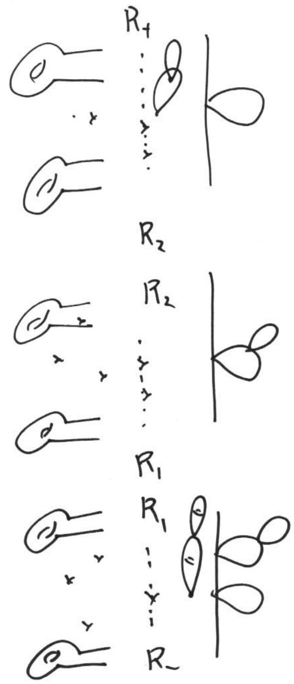

Figure 2.6

Theorem 2.15. *$\mathcal{M}((M,\mathcal{E}),L;R_-,R_+;E)$ is compact and Hausdorff.*

The compactness follows from Uhlenbeck compactness in gauge theory side ([Uh1], [Uh2]), Gromov compactness in symplectic geometry side ([Gr]. See [Ye] for the case of moduli space of pseudo-holomorphic disks and [FOn, Theorem 11.1] for the way to adapt the proof to the case when a particular version of the topology is used in the pseudo-holomorphic curve side, that is, the stable map topology.) and [Fu5, Theorems 1.6, 1.7 and 1.8]. The Hausdorff-ness can be proved in the same way as the proof of [FOn, Lemma 10.4].

To describe the boundary of $\mathcal{M}((M,\mathcal{E}),L;R_-,R_+;E)$ we include boundary marked points.

Definition 2.16. We consider $(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z})$ such that:

(1) $\mathfrak{A}$ satisfies Definition 2.9 (1).
(2) $\mathfrak{z}$ satisfies Definition 2.9 (2).
(3) $\mathfrak{w}$ satisfies Definition 2.9 (3).
(4) $\vec{z}=(z_1,\dots,z_k)$. z_i lies on $\partial\Sigma$. Namely it lies either on $\{1\}\times\mathbb{R}$ or on the boundary of one of the disk components. None of z_i is a nodal point and $z_i\neq z_j$ for $i\neq j$. $(z_1,\dots,z_k)$ respects counter clockwise orientation of $\partial\Omega$.
(5) Ω satisfies Condition 2.7.
(6) (Ω,u) satisfies Condition 2.8 except (3), which we replace by the stability of $(\Omega,u,\vec{z})$. Namely the set of all maps $v:\Omega\to\Omega$ satisfying the next three conditions is a finite set.
    (a) v is a homeomorphism and is holomorphic on each of the irreducible components.

(b) v is the identity map on $(0,1] \times \mathbb{R} \subseteq \Omega$.
(c) $u \circ v = u$.
(d) $v(z_i) = z_i$, $i = 1, \dots, k$.

(7) Definition 2.9 (6)(7)(8) hold.

We can define equivalence among such objects by modifying Definition 2.11 in an obvious way.

Let $\overset{\circ}{\widetilde{\mathcal{M}}}_k((M,\mathcal{E}), L; R_-, R_+; E)$ be the set of equivalence classes of such objects. We can define a topology on it by modifying Definition 2.12 in an obvious way.

Let $\overset{\circ}{\mathcal{M}}_k((M,\mathcal{E}), L; R_-, R_+; E)$ be its quotient space by $\mathbb{R}$ action.

Replacing (2.23) by

$$\begin{aligned}&\overset{\circ}{\mathcal{M}}_{k_0}((M,\mathcal{E}), L; R_-, R_1; E_1) \times_{R_1} \overset{\circ}{\mathcal{M}}_{k_1}((M,\mathcal{E}), L; R_1, R_2; E_2) \times_{R_2} \dots \\ &\cdots \times_{R_{\ell-1}} \overset{\circ}{\mathcal{M}}_{k_{\ell-1}}((M,\mathcal{E}), L; R_{\ell-1}, R_\ell; E_{\ell-1}) \\ &\times_{R_\ell} \mathcal{M}_{k_\ell}((M,\mathcal{E}), L; R_\ell, R_+; E_\ell),\end{aligned} \tag{2.25}$$

where $k_0 + k_1 + \cdots + k_\ell = k$, $E_0 + \cdots + E_\ell = E$, and $R_1, \dots, R_\ell$ are connected components of $R(M) \times_{R(\Sigma)} \tilde{L}$, we obtain $\mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E)$.

The space $\mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E)$ is compact and Hausdorff.

There exists an evaluation map

$$\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_k) : \mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E) \to L^k \tag{2.26}$$

other than ev_- and ev_+. (See (2.22) for $\mathrm{ev}_\pm$.) If $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u, \vec{z})$ is an element of $\overset{\circ}{\widetilde{\mathcal{M}}}_k((M,\mathcal{E}), L; R_-, R_+; E)$ then, by definition

$$\mathrm{ev}_i([\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u, \vec{z}]) = u(z_i). \tag{2.27}$$

We use the compactified moduli space of pseudo-holomorphic disks in the next theorem. Let $\beta \in H_2(R(\Sigma), L; \mathbb{Z})$. We denote by $\mathcal{M}_{k+1}(L;\beta)$ the compactified moduli space of pseudo-holomorphic disks bounding L with $k+1$ boundary marked points and of homology class β. (See [FOOO1, Definition 2.1.27] for its precise definition.) We have an evaluation map

$$\mathrm{ev} = (\mathrm{ev}_0, \dots, \mathrm{ev}_k) : \mathcal{M}_{k+1}(L;\beta) \to L^{k+1}. \tag{2.28}$$

We put

$$\mathcal{M}_{k+1}(L;E) = \bigcup_{\beta;\omega(\beta)=E} \mathcal{M}_{k+1}(L;\beta). \tag{2.29}$$

Theorem 2.17. *The space $\mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E)$ has a virtual fundamental chain $[\mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E)]$ such that its boundary $\partial[\mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E)]$ is a sum of the virtual fundamental chains of the following two types of spaces.*

(1) *The fiber product*

$$\mathcal{M}_{k_1}((M,\mathcal{E}), L; R_-, R; E_1) \times_R \mathcal{M}_{k_2}((M,\mathcal{E}), L; R, R_+; E_2),$$

where $E_1 + E_2 = E$ and $k_1 + k_2 = k$. We use ev_+ for the first factor and ev_- for the second factor to define the above fiber product.

(2) *The fiber product:*

$$\mathcal{M}_{k_1}((M,\mathcal{E}), L; R_-, R; E_1)_{\mathrm{ev}_i} \times_{\mathrm{ev}_0} \mathcal{M}_{k_2}(L, E_2),$$

where $E_1 + E_2 = E$, $k_1 + k_2 = k + 1$, $i = 1, \ldots, k_1$. *The fiber product is taken over* L.

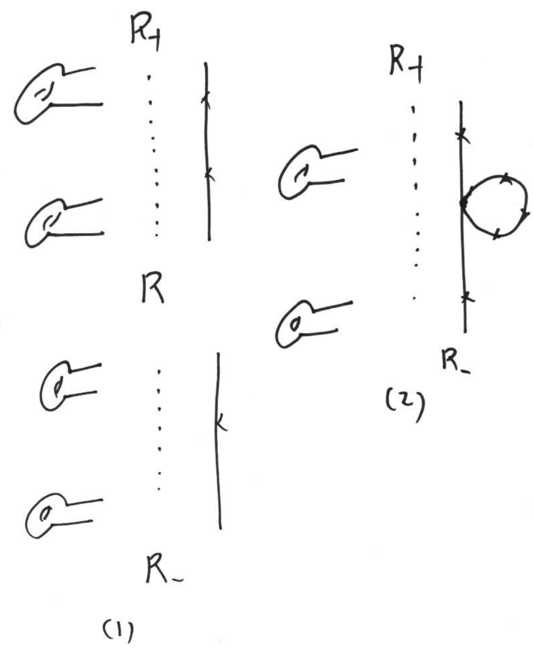

Figure 2.7

Remark 2.18. We do *not* claim that $\mathcal{M}_k((M, \mathcal{E}), L; R_-, R_+; E)$ has a Kuranishi structure in the sense of [FOn] or [FOOO1]. This is because our moduli space is a mixture of gauge theory and of pseudo-holomorphic curve. It is well known among the specialists that Uhlenbeck compactification of the moduli space of ASD connections does *not* carry a Kuranishi structure. In Donaldson theory, people, especially Donaldson, used the fact that Uhlenbeck compactification has a stratification for which the top stratum has dimension higher by 2 or more compared to the other strata, in order to define its fundamental class. This fundamental class is nothing but the Donaldson invariant ([D3]).

In our situation, gauge theory is mixed up with symplectic geometry (pseudo-holomorphic curve). It seems to the author that the best way to work out transversality issue is to use virtual technique, eg. Kuranishi structure.

So we need to work out some generalization of the notion of Kuranishi structure which has certain 'singularity' at the set of codimension equal to or greater than 2. We will provide the detail of the framework for such generalization in [Fu8] (or in certain separate paper).

We also remark that in our situation where $R(\Sigma)$ is monotone, we can work over $\mathbb{Z}_2$ coefficient. (See [FOOO5].) On the other hand, the author has no doubt that one can work out the whole story over $\mathbb{Z}$ by carefully studying orientation and sign.

We also remark that, if we restrict ourselves to the proof of Corollary 1.2, we can avoid using virtual technique. The reason is that the situation we need to study for such a purpose is monotone. Especially the Lagrangian submanifold $R(M)$ is

monotone if it is embedded. We will work out this part of the story in detail in [Fu7].

We use virtual fundamental chain in Theorem 2.17, to define the operation

$$\mathfrak{n}_{k,E} : C(R(M) \times_{R(\Sigma)} L; \mathbb{Z}_2) \otimes C(L; \mathbb{Z}_2)^{k\otimes} \to C(R(M) \times_{R(\Sigma)} L; \mathbb{Z}_2), \tag{2.30}$$

by

$$\begin{aligned} &\langle \mathfrak{n}_{k,E}(y_-; x_1, \dots, x_k), y_+\rangle \\ &= \#\big([\mathcal{M}_k((M,\mathcal{E}), L; R_-, R_+; E)]_{(\mathrm{ev}_-, \mathrm{ev}_1, \dots, \mathrm{ev}_k, \mathrm{ev}_+)} \times \\ &\qquad\qquad\qquad\qquad (y_- \times x_1 \times \dots \times x_k \times y_+)\big). \end{aligned} \tag{2.31}$$

Here $\langle \cdot, \cdot \rangle$ in the left hand side is the Poincaré duality of $R(M) \times_{R(\Sigma)} L$, and $\#$ is the parity of the order of the set $\in \mathbb{Z}_2$. The symbol $C(\cdot)$ denotes an appropriate chain model of the homology group. Then Theorem 2.17 will imply the next formula:

$$\begin{aligned} 0 =&(\partial \circ n_{E,k})(y; x_1, \dots, x_k) + (n_{E,k} \circ \partial)(y; x_1, \dots, x_k) \\ &+ \sum \mathfrak{n}_{k_2,E_2}(\mathfrak{n}_{k_1,E_1}(y; x_1, \dots, x_{k_1}); x_{k_1+1}, \dots, x_k) \\ &+ \sum \mathfrak{n}_{k_1}(y; x_1, \dots, \mathfrak{m}_{k_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k). \end{aligned} \tag{2.32}$$

Here the sum in the second line is taken over E_1, E_2, k_1, k_2 satisfying $E_1 + E_2 = E$ and $k_1 + k_2 = k$. The sum in the third line is taken over E_1, E_2, k_1, k_2, i satisfying $E_1 + E_2 = E$ and $k_1 + k_2 = k + 1$ and $i = 1, \dots, k_1$.

Note the second line of (2.32) corresponds to Theorem 2.17 (1) and the third line of (2.32) corresponds to Theorem 2.17 (2). Therefore, in case we take de Rham theory as our chain model, the formula (2.32) follows from Theorem 2.17 together with Stokes' formula ([FOOO8, Theorem 8.11]) and composition formula ([FOOO8, Theorem 10.20]). The way to use Stokes' formula and composition formula to prove a formula like (2.32) will be explained in detail in [FOOO9].

However, as we mentioned in Remark 2.18, our moduli space do not carry a genuine Kuranishi structure but has a singularity of codimension 2 or higher. So it seems not easy to use de Rham model. We will explain in detail the particular chain model we use for our purpose and the way how we use it to prove (2.32) in detail in [Fu8], or in a separate paper.

Definition 2.19. We define

$$\begin{aligned} \mathfrak{n}_0 &= \partial + \sum_{E \geq 0} T^E \mathfrak{n}_{0,E}, \\ \mathfrak{n}_k &= \sum_{E \geq 0} T^E \mathfrak{n}_{k,E}, \qquad k \geq 1. \end{aligned} \tag{2.33}$$

We thus obtained a system of operations:

$$\mathfrak{n}_k : C(R(M) \times_{R(\Sigma)} L; \Lambda_0^{\mathbb{Z}_2}) \otimes C(L; \Lambda_0^{\mathbb{Z}_2})^{k\otimes} \to C(R(M) \times_{R(\Sigma)} L; \Lambda_0^{\mathbb{Z}_2}). \tag{2.34}$$

Note we put

$$CF(R(M), L) = C(R(M) \times_{R(\Sigma)} L; \Lambda_0^{\mathbb{Z}_2}), \quad CF(L) = C(L; \Lambda_0^{\mathbb{Z}_2}).$$

Theorem 2.20. *The system of operations $\mathfrak{n}_k$, $k = 0, 1, 2, \dots$ defines a structure of a filtered A_∞ right module on $CF(R(M), L)$ over the filtered A_∞ algebra $(CF(L), \{\mathfrak{m}_k \mid k = 0, 1, 2, \dots\})$ defined in* [FOOO5]. *Namely the equality (2.7) holds.*

Remark 2.21. (1) Because of the problem of 'running out' described in [FOOO2, Section 7.2.3] it is actually difficult to construct all the operations $\mathfrak{n}_E$ at once. So we need to construct $\mathfrak{n}_E$ for $E \le E_0$ and use homological algebra to take homotopy limit. Since this argument is now well established in [FOOO2, Section 7] (see also [FOOO9]) we do not repeat it here for simpicity.

(2) To prove the convergence of the right hand side of (2.33) in T-adic topology we need the next Theorem 2.22 which is slightly stronger than Theorem 2.15. The proof of Theorem 2.22 is the same as that of Theorem 2.15.

Theorem 2.22. *For any E_0, the union of the moduli spaces $\mathcal{M}((M,\mathcal{E}),L;R_-,R_+;E)$ for $E \le E_0$ is compact.*

We thus explained the outline of the construction of the filtered A_∞ right module $(CF(R(M),L),\{\mathfrak{n}_k\})$ over the filtered A_∞ algebra $(CF(L),\{\mathfrak{m}_k \mid k=0,1,2,\dots\})$ in case L is embedded.

We can extend this construction to the construction of the filtered A_∞ functor $\mathcal{HF}_{(M,\mathcal{E}_M)} : \mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$. Namely we can define a series of operations:

$$\mathfrak{n}_k : CF(R(M),L_1) \otimes \bigotimes_{i=1}^{k-1} CF(L_i,L_{i+1}) \to CF(R(M),L_k) \tag{2.35}$$

which satisfies the same relation (2.7). The proof is similar to the proof of Theorem 2.20 using the moduli space of objects shown in the figure below. See also [Fu4, Theorem 4.8].

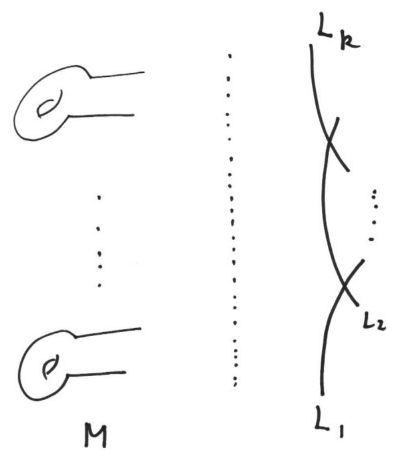

Figure 2.8

We next describe the way how to generalize the construction to the case when L is immersed. Let $i_L : \tilde{L} \to R(\Sigma)$ be a Lagrangian immersion with image L. We decompose the fiber product $\tilde{L} \times_{R(M)} \tilde{L}$ into connected components and write

$$\tilde{L} \times_{R(\Sigma)} \tilde{L} = \bigcup_{i \in I(L)} \hat{L}_i. \tag{2.36}$$

Note in our situation where the self-intersection of L is transversal, $\hat{L}_i$ is either a connected component of L or one point consisting $(p,q) \in \tilde{L}^2$ with $p \neq q$, $i_L(p) = i_L(q)$. In the later case we call $\hat{L}_i = \{(p,q)\}$ *switching component.*

We also decompose

$$R(M) \times_{R(\Sigma)} \tilde{L} = \bigcup_{i \in I(R(M),L)} R_i. \tag{2.37}$$

Now we modify Definition 2.16 as follows. Definition 2.23 below is mostly the same as Definition 2.16. The difference lies in items (6), (7) and (11).

Definition 2.23. Let R_+ and R_- are one of the connected components R_i ($i \in I(R(M),L)$). Let $\vec{i} = (i(1),\dots,i(k))$ where $i(1),\dots,i(k) \in I(L)$. Here $I(L)$ (resp. $I(R(M),L)$) is as in (2.36) (resp. (2.37)). We put $|\vec{i}| = k$. We consider $(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z})$ such that:

(1) $\mathfrak{A}$ satisfies Definition 2.9 (1).
(2) $\mathfrak{z}$ satisfies Definition 2.9 (2).
(3) $\mathfrak{w}$ satisfies Definition 2.9 (3).
(4) $\vec{z} = (z_1,\dots,z_k)$. z_i lies on $\partial\Sigma$. Namely it lies either on $\{1\} \times \mathbb{R}$ or on the boundary of one of the disk components. None of z_i is a nodal point and $z_i \neq z_j$ if $i \neq j$. $(z_1,\dots,z_k)$ respects counter clockwise orientation of $\partial\Omega$.
(5) Ω satisfies Condition 2.7.
(6) There exists $\gamma : \partial\Sigma \setminus \{z_1,\dots,z_k\} \to \tilde{L}$ such that $u(z) = i_L(\gamma(z))$ on $\partial\Sigma \setminus \{z_1,\dots,z_k\}$.
(7) For $j = 1,\dots,k$ the following holds.

$$(\lim_{z\uparrow z_j} \gamma(z), \lim_{z\downarrow z_j} \gamma(z)) \in R_{i(j)}. \tag{2.38}$$

Here $z \uparrow z_j$ is the limit when $z \in \partial\Omega$ converges to z_j while moving to the counter clockwise direction. $z \downarrow z_j$ is the limit when $z \in \partial\Omega$ converges to z_j while moving to the clockwise direction.

If $\hat{L}_{i(j)}$ is not a switching component then (2.38) means that γ extends to a continuous map at z_j. If $\hat{L}_{i(j)}$ is a switching component consisting of the point (p,q), then (2.38) means $\lim_{z\uparrow z_j} \gamma(z) = p$, $\lim_{z\downarrow z_j} \gamma(z) = q$.
(8) (Ω,u) satisfies Condition 2.8 (2).
(9) We replace Condition 2.8 (3) by the stability of $(\Omega,u,\vec{z})$. Namely the set of all maps $v : \Omega \to \Omega$ satisfying the next four conditions is a finite set.
 (a) v is a homeomorphism and is holomorphic on each of the irreducible components.
 (b) v is the identity map on $(0,1] \times \mathbb{R} \subseteq \Omega$.
 (c) $u \circ v = u$.
 (d) $v(z_i) = z_i$, $i = 1,\dots,k$.
(10) Definition 2.9 (6)(7) hold.
(11) Let a_- and a_+ be as in the conclusion of Theorem 2.4. Then

$$([a_-], \lim_{z\downarrow -\infty} \gamma(z)) \in R_-, \qquad ([a_+], \lim_{z\uparrow +\infty} \gamma(z)) \in R_+. \tag{2.39}$$

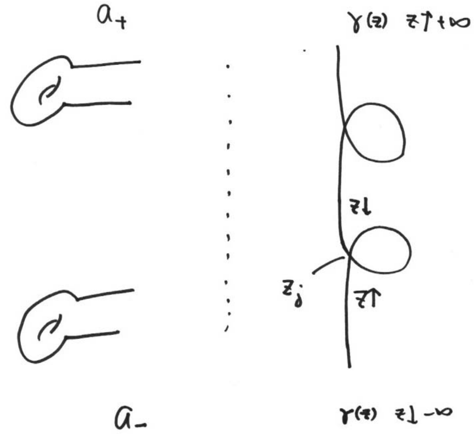

Figure 2.9

We can define equivalence among such objects by modifying Definition 2.11 in an obvious way.

Let $\widetilde{\overset{\circ}{\mathcal{M}}}((M,\mathcal{E}),L;R_-,R_+;\vec{i};E)$ be the set of equivalence classes of such objects. We can define a topology on it by modifying Definition 2.12 in an obvious way.

We put

$$\widetilde{\overset{\circ}{\mathcal{M}}}_k((M,\mathcal{E}),L;R_-,R_+;E) = \bigcup_{\vec{i};|\vec{i}|=k} \widetilde{\overset{\circ}{\mathcal{M}}}((M,\mathcal{E}),L;R_-,R_+;\vec{i};E)$$

Let $\overset{\circ}{\mathcal{M}}_k((M,\mathcal{E}),L;R_-,R_+;E)$ be its quotient space by $\mathbb{R}$ action.

By taking the union (2.25) we obtain $\mathcal{M}_k((M,\mathcal{E}),L;R_-,R_+;E)$.

$\mathcal{M}_k((M,\mathcal{E}),L;R_-,R_+;E)$ is a compact Hausdorff space.

There exists an evaluation map

$$\mathrm{ev} = (\mathrm{ev}_1,\dots,\mathrm{ev}_k) : \mathcal{M}_k((M,\mathcal{E}),L;R_-,R_+;E) \to (\tilde{L}\times_{R(M)}\tilde{L})^k. \tag{2.40}$$

If $(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z})$ is an element of $\widetilde{\overset{\circ}{\mathcal{M}}}((M,\mathcal{E}),L;R_-,R_+;\vec{i};E)$ then, by definition

$$\mathrm{ev}_j([\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z}]) = (\lim_{z\uparrow z_j}\gamma(z), \lim_{z\downarrow z_j}\gamma(z)) \in R_{i(j)}. \tag{2.41}$$

We also define the evaluation maps

$$\begin{aligned} \mathrm{ev}_-(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z}) &= ([a_-], \lim_{z\downarrow -\infty}\gamma(z)) \in R_-, \\ \mathrm{ev}_+(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z}) &= ([a_+], \lim_{z\uparrow +\infty}\gamma(z)) \in R_+. \end{aligned} \tag{2.42}$$

Now in the same way as (2.31) we obtain

$$\mathfrak{n}_{k,E} : C(R(M)\times_{R(\Sigma)}L;\mathbb{Z}_2)\otimes C(\tilde{L}\times_{R(M)}\tilde{L};\mathbb{Z}_2)^{k\otimes} \to C(R(M)\times_{R(\Sigma)}\tilde{L};\mathbb{Z}_2). \tag{2.43}$$

Then we apply Definition 2.19 to obtain

$$\mathfrak{n}_k : CF(R(M),L)\otimes CF(L)^{k\otimes} \to CF(R(M),L). \tag{2.44}$$

Theorem 2.20 still holds in our immersed case.

We thus described the construction of right filtered A_∞ module $(CF(R(M),L),\{\mathfrak{n}_k\})$. It is straight forward to generalize this construction to the construction of filtered A_∞ functor $\mathcal{HF}_{(M,\mathcal{E}_M)} : \mathscr{FUK}(R(\Sigma)) \to \mathscr{CH}$. Namely we can construct the operations

$$\mathfrak{n}_k : CF(R(M),L_0)\otimes\bigotimes_{i=1}^{k} CF(L_{i-1},L_i) \to CF(R(M),L_k) \tag{2.45}$$

which satisfies the same relation (2.7). In fact we may regard the union $L_0\cup\dots\cup L_k$ as a single immersed Lagrangian submanifold and can apply Definition 2.23 etc.

We have thus completed the sketch of the proof of Theorem 1.3.

Remark 2.24. In case L is an embedded and monotone Lagrangian submanifold of $R(\Sigma)$ the construction of $HF((M,\mathcal{E});L)$ is carried out in detail by [SaWe]. More precisely they did the case when the restriciton of $\mathcal{E}$ to Σ is a trivial $SU(2)$ bundle. This case is harder than the case we study here.

3. Floer theoretical unobstructed-ness of the moduli space of flat connections on 3-manifolds with boundary

In this section we prove Theorem 1.1 (1). The main idea of its proof is an algebraic result, Proposition 3.5 below. To state it we introduce certain notions.

Definition 3.1. (Compare [FOOO1, Condition 3.1.6]) A discrete submonoid G of $\mathbb{R}_{\ge 0}$ is a discrete subset $G\subset\mathbb{R}_{\ge 0}$ containing 0 such that $\lambda_1,\lambda_2\in G$ implies $\lambda_1+\lambda_2\in G$. Hereafter we say *discrete monoid* in place of discrete submonoid of $\mathbb{R}_{\ge 0}$ for simplicity.

Definition 3.2. ([FOOO1, Definition 3.2.26 etc.]) Let $\overline{C}$, $\overline{C}_i$ $(i=1,2)$ be $\mathbb{Z}_2$ vector spaces and C (resp. C_i) the completion of $\overline{C}\otimes\Lambda_0^{\mathbb{Z}_2}$ (resp. of $\overline{C}_i\otimes\Lambda_0^{\mathbb{Z}_2}$) with respect to the T-adic topology. Let G be a discrete monoid.

(1) An element x of C is said to be *G-gapped* if there exists $x_\lambda\in\overline{C}$ for each $\lambda\in G$ such that
$$x=\sum_{\lambda\in G}T^\lambda x_\lambda.$$
(2) A $\Lambda_0^{\mathbb{Z}_2}$ module homomorpism $\varphi : C_1\to C_2$ is said to be *G-gapped* if there exists a $\mathbb{Z}_2$ linear maps $\varphi_\lambda:\overline{C}_1\to\overline{C}_2$ such that
$$\varphi=\sum_{\lambda\in G}T^\lambda\varphi_\lambda.$$
(3) A filtered A_∞ algebra (resp. a filtered A_∞ module) is said to be *G-gapped* if its structure maps $\mathfrak{m}_k$ (resp. $\mathfrak{n}_k$) are all G-gapped.

Definition 3.3. Let $(C,\{\mathfrak{m}_k\})$ be a G-gapped filtered A_∞ algebra and $(D,\{\mathfrak{n}_k\})$ be a G-gapped right filtered A_∞ module over $(C,\{\mathfrak{m}_k\})$.

We say $\mathbf{1}\in D$ is a *cyclic element*[4] if the following holds

(1) The map $C\to D$ which sends x to $\mathfrak{n}_2(\mathbf{1};x)$ is an $\Lambda_0^{\mathbb{Z}_2}$ module isomorphism $C\to D$.
(2) $\mathfrak{n}_0(\mathbf{1})\equiv 0 \mod \Lambda_+^{\mathbb{Z}_2}$.

[4]The word cyclic element seems to be a standard one for an object satisfying a condition such as (1). We remark that the notion of cyclic element has no relation to the cyclic symmetry of the filtered A_∞ algebra associated to a Lagrangian submanifold.

We also recall:

Definition 3.4. Let C be a filtered A_∞ algebra. An element $b \in C \otimes_{\Lambda_0^{\mathbb{Z}_2}} \Lambda_+^{\mathbb{Z}_2}$ is said to be a *bounding cochain* if

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) = 0. \tag{3.1}$$

Note the left hand side converges in T-adic topology, since $b \equiv 0 \mod \Lambda_+^{\mathbb{Z}_2}$.

Proposition 3.5. *Let $(C, \{\mathfrak{m}_k\})$ be a G-gapped filtered A_∞ algebra and $(D, \{\mathfrak{n}_k\})$ a G-gapped right filtered A_∞ module over $(C, \{\mathfrak{m}_k\})$. Suppose $\mathbf{1} \in D$ is a cyclic element, which is G-gapped.*

Then there exists a unique G-gapped bounding cochain b of $(C, \{\mathfrak{m}_k\})$ such that

$$d^b(\mathbf{1}) = 0. \tag{3.2}$$

Note we defined d^b by

$$d^b(y) = \sum_{k=0}^{\infty} \mathfrak{n}_k(y; b, \dots, b).$$

Proof. We first prove the uniqueness. Let $G = \{\lambda_i \mid i = 0, 1, 2, \dots\}$ where $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$. We put

$$\mathbf{1} = \sum_{i=0}^{\infty} T^{\lambda_i} \mathbf{1}_i, \qquad b = \sum_{i=1}^{\infty} T^{\lambda_i} b_i$$
$$\mathfrak{m}_k = \sum_{i=0}^{\infty} T^{\lambda_i} \mathfrak{m}_{k,i}, \qquad \mathfrak{n}_k = \sum_{i=0}^{\infty} T^{\lambda_i} \mathfrak{n}_{k,i}.$$

according to the definition of G-gapped-ness. (Note the coefficient of T^{λ_0} $(\lambda_0 = 0)$ of b is 0 since $b \in C \otimes_{\Lambda_0^{\mathbb{Z}_2}} \Lambda_+^{\mathbb{Z}_2}$.)

We calculate the coefficient of T^{λ_n} of the equation (3.2) and obtain

$$\mathfrak{n}_{1,0}(\mathbf{1}_0; b_n) + \sum \mathfrak{n}_{k,m}(\mathbf{1}_{n_0}; b_{n_1}, \dots, b_{n_k}) = 0. \tag{3.3}$$

Here the second term is the sum of all k, m, $n_0, n_1, \dots, n_k$ such that

$$\lambda_n = \lambda_m + \lambda_{n_0} + \sum_{i=1}^{k} \lambda_{n_i} \tag{3.4}$$

except the case $k = 1$, $m = 0$, $n_0 = 0$, $n_1 = n$. (The case we exclude here corresponds to the first term.) Note if k, m, $n_0, n_1, \dots, n_k$ satisfy (3.4) then $n_i \le n$ for $i = 0, \dots, k$. Moreover $n_i < n$ unless $k = 1$, $m = 0$, $n_0 = 0$, $n_1 = n$.

Therefore we can solve (3.3) and obtain b_n uniquely by induction on n. (Here we use Definition 3.3 (1).)

Thus we proved that there exists a unique G-Gapped element $b \in C \otimes_{\Lambda_0^{\mathbb{Z}_2}} \Lambda_+^{\mathbb{Z}_2}$ satisfying (3.2).

It remains to prove that this element b satisfies the Maurer-Cartin equation (3.1). We will prove

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \dots, b) \equiv 0 \mod T^{\lambda_c} \tag{3.5}$$

by induction on $c \in \mathbb{Z}_+$. We assume (3.5) for $c \le n-1$ and will prove the case $c = n$ below.

We put $\partial = \mathfrak{n}_{1,0}$. (By (2.32) we have $\partial \circ \partial = 0$.) Using (2.32) and Definition 3.3 (2) we have

$$\partial \mathfrak{n}_{1,0}(\mathbf{1}_0; x) + \mathfrak{n}_{1,0}(\mathbf{1}_0; \partial x) = 0 \tag{3.6}$$

for $x \in \overline{C}$.

We next consider $\partial \mathfrak{n}_{1,0}(\mathbf{1}_0; b_n)$. Using (3.3) we find

$$\partial(\mathfrak{n}_{1,0}(\mathbf{1}_0; b_n)) = \sum \partial(\mathfrak{n}_{k,m}(\mathbf{1}_{n_0}; b_{n_1}, \dots, b_{n_k})).$$

We calculate the right hand side using (2.32) to obtain:

$$\begin{aligned}
&\sum \mathfrak{n}_{k_1,m_1}(\mathfrak{n}_{k_2,m_2}(\mathbf{1}_{n_0}; b_{n_1}, \dots, b_{n_{k_2}}), \dots, b_{n_k}) \\
&+ \sum \mathfrak{n}_{k_1,m_1}(\mathbf{1}_{n_0}; b_{n_1}, \dots, \mathfrak{m}_{k_2,m_2}(b_{n_{i+1}}, \dots, b_{n_{i+k_2}}), \dots, b_{n_k}) \\
&+ \sum \mathfrak{n}_{k,m}(\mathbf{1}_{n_0}; b_{n_1}, \dots, \partial b_{n_j}, \dots, b_{n_k}).
\end{aligned} \tag{3.7}$$

Here the sum in the first line is taken over k_1, k_2, m_1, m_2, $n_0, \dots, n_k$ such that $k_1 + k_2 = k$ and $\lambda_n = \lambda_{m_1} + \lambda_{m_2} + \lambda_{n_0} + \sum_{i=1}^k \lambda_{n_i}$, except $(k_1, m_1) = (0,0)$.

The sum in the second line is taken over k_1, k_2, m_1, m_2, $n_0, \dots, n_k$ such that $k_1 + k_2 = k+1$ and $\lambda_n = \lambda_{m_1} + \lambda_{m_2} + \lambda_{n_0} + \sum_{i=1}^k \lambda_{n_i}$, except $m_2 = 0$, $k_2 = 1$. (The excluded case corresponds to the third line.)

The sum in the third line is taken over k, m, j, $n_0, \dots, n_k$ such that $j = 1, \dots, k$ and $\lambda_n = \lambda_m + \lambda_{n_0} + \sum_{i=1}^k \lambda_{n_i}$, except $n_0 = 0$, $k = 1$, $m = 0$. We exclude this case since it is excluded in the second term of (3.3).

Note the first line of (3.7) vanishes because of the equality (3.2) and $(k_1, m_1) \neq (0,0)$.

By using induction hypothesis (3.5) for $c \le n-1$, the sum of the second and third lines cancel each other except the sum

$$\sum \mathfrak{n}_{0,1}(\mathbf{1}_0; \mathfrak{m}_{k,m}(b_{n_1}, \dots, b_{n_k}))$$

which is taken over $k, m, n_1, \dots, n_k$ such that $\lambda_n = \lambda_m + \sum_{i=1}^k \lambda_{n_i}$. (In fact this sum could be canceled with $\mathfrak{n}_{0,1}(\mathbf{1}_0; \partial b_n)$. However this is the case excluded in the third line.)

Thus we have

$$\mathfrak{n}_{1,0}(\mathbf{1}_0; \partial b_n) = \partial \mathfrak{n}_{1,0}(\mathbf{1}_0; b_n) = \sum \mathfrak{n}_{0,1}(\mathbf{1}_0; \mathfrak{m}_{k,m}(b_{n_1}, \dots, b_{n_k})).$$

Using Definition 3.3 (1) it implies

$$\partial b_n = \sum \mathfrak{m}_{k,m}(b_{n_1}, \dots, b_{n_k}).$$

It implies (3.5) for $c = n$. The proof of Proposition 3.5 is now complete. □

Remark 3.6. Suppose $(C, \{\mathfrak{m}_k\})$ is a filtered A_∞ algebra and has a unit $\mathbf{e}$. Then by defining

$$\mathfrak{n}_{k_1,k_2}(x_1, \dots, x_{k_1}; y; z_1, \dots, z_{k_2}) = \mathfrak{m}_{k_1+k_2+1}(x_1, \dots, x_{k_1}, y, z_1, \dots, z_{k_2})$$

C can be regarded as a filtered A_∞ bimodule over $(C, \{\mathfrak{m}_k\})$-$(C, \{\mathfrak{m}_k\})$. (See [FOOO1, Example 3.7.6].) Moreover $\mathbf{e} = \mathbf{1}$ satisfies Definition 3.3 (1)(2).

However in case $\mathfrak{m}_0 \neq 0$ the operations

$$\mathfrak{n}_k(y; x_1, \dots, x_k) = \mathfrak{m}_{k+1}(y, x_1, \dots, x_k)$$

do *not* define a structure of filtered A_∞ right module on C over $(C, \{\mathfrak{m}_k\})$. In fact we have

$$\begin{aligned}&\sum_{\ell=0}^{k} \mathfrak{n}_{k-\ell+1}(\mathfrak{n}_\ell(y; x_1, \dots, x_\ell); x_{\ell+1}, \dots, x_k)\\ &+ \sum_{0\le \ell\le m\le k} \mathfrak{n}_{k-m+\ell+1}(y; x_1, \dots, \mathfrak{m}_{m-\ell}(x_\ell, \dots, x_{m-1}), \dots, x_k)\\ &= \mathfrak{n}_{1,k}(\mathfrak{m}_0(1); y, x_1, \dots, x_k).\end{aligned}$$

Therefore we can *not* apply Proposition 3.5 in this situation.

Now we apply Proposition 3.5 to our geometric situation and prove Theorem 1.1 (1). Let M be a 3 manifold with boundary Σ and $\mathcal{E}_M$ an $SO(3)$ bundle on M such that the 2nd Stiefel-Whitney class $w_2(\mathcal{E}_\Sigma)$ of the restriction $\mathcal{E}_\Sigma$ of $\mathcal{E}_M$ to Σ is the fundamental class $[\Sigma]$. Let $i_{R(M)} : R(M) \to R(\Sigma)$ be the Lagrangian immersion, where $R(M)$ (resp. $R(\Sigma)$) is the set of gauge equivalence classes of flat connections of $\mathcal{E}_M$ (resp. $\mathcal{E}_\Sigma$).

In the previous section, we associate a right filtered A_∞ module $CF(M, L)$ over $(CF(L), \{\mathfrak{m}_k\})$ to a Lagrangian immersion $i_L : \tilde L \to L \subseteq R(\Sigma)$. These filtered A_∞ algebra and filtered A_∞ module are G-gapped for certain discrete monoid G by Gromov-Uhlenbeck compactness and Theorem 2.22.

We consider its special case where $\tilde L = R(M)$. In this case the underlying $\Lambda_0^{\mathbb{Z}_2}$ module of $(CF(M, L), \{\mathfrak{n}_k\})$ coincides with $(CF(R(M), R(M)), \{\mathfrak{m}_k\})$. Namely both are $C(R(M) \times_{R(\Sigma)} R(M); \Lambda_0^{\mathbb{Z}_2})$. We remark

$$R(M)\times_{R(\Sigma)} R(M) \cong R(M)\sqcup\{(p, q) \in R(M)^2 \mid p \ne q, i_{R(M)}(p) = i_{R(M)}(q)\}. \tag{3.8}$$

Therefore

$$C(R(M) \times_{R(\Sigma)} R(M); \Lambda_0^{\mathbb{Z}_2}) = C(R(M); \Lambda_0^{\mathbb{Z}_2}) \oplus \bigoplus_{(p,q)} \Lambda_0^{\mathbb{Z}_2}[(p, q)]$$

where the direct sum in the second component is taken over $\{(p, q) \in R(M)^2 \mid p \ne q, i(p) = i(q)\}$.

Definition 3.7. We take $\mathbf{1}_M \in C(R(M) \times_{R(\Sigma)} R(M); \Lambda_0^{\mathbb{Z}_2})$ to be the fundamental cycle of $R(M)$ in $C(R(M); \Lambda_0^{\mathbb{Z}_2})$.

Now we observe:

Lemma 3.8. $\mathbf{1}_M$ *is a cyclic element of the right filtered* A_∞ *module* $(C(R(M)\times_{R(\Sigma)} R(M); \Lambda_0^{\mathbb{Z}_2}), \{\mathfrak{n}_k\})$ *over* $(C(R(M) \times_{R(\Sigma)} R(M); \Lambda_0^{\mathbb{Z}_2}), \{\mathfrak{m}_k\})$.

Proof. We decompose

$$\mathfrak{n}_k = \sum T^{\lambda_i} \mathfrak{n}_{k,i}$$

accoding to the definition of G-gapped-ness. Then by definition $\mathfrak{n}_{k,0}$ is defined by using the moduli spaces $\mathcal{M}_k((M, \mathcal{E}), L; R_-, R_+; 0)$ consisting of the solutions of (2.12), (2.14) with zero energy. Such solution is necessary of the form $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ where $\mathfrak{A}$ is a flat connection on $M \times \mathbb{R}$, u is a constant map and $\Omega = \Sigma \times (0, 1] \times \mathbb{R}$. It follows that

$$\mathfrak{n}_{1,0}(\mathbf{1}_M; x) = x$$

for any $x \in C(R(M) \times_{R(\Sigma)} R(M); \Lambda_0^{\mathbb{Z}_2})$. Definition 3.3 (1) follows immediately.

Definition 3.3 (2) follows from the fact that $\mathfrak{n}_0$ is congruent to the classical boundary operator modulo $\Lambda_+^{\mathbb{Z}_2}$. □

Proposition 3.5 and Lemma 3.8 immediately imply the next theorem.

Theorem 3.9. *The immersed Lagrangian submanifold $R(M) \to R(\Sigma)$ is unobstructed. Moreover we can chose the bounding cochain b_M uniquely so that*

$$d^{b_M}(\mathbf{1}_M) = 0. \tag{3.9}$$

Theorem 1.1 (1) follows from Theorem 3.9 except the statement that the gauge equivalence class of b_M is independent of the choices. Here the choices are perturbation to define filtered A_∞ algebra and filtered A_∞ module involved in the construction and the metric on M etc.. One can prove this independence by using a cobordism argument, which have been used extensively in various related situations. (The one which is closest to the present situation is [Fu4, Sections 5,6,7].) So we can safely omit it in this paper and postpone its detail to [Fu8].

Theorem 1.1 (3) follows from the next proposition.

Proposition 3.10. *If $R(M)$ is an embedded Lagrangian submanifold then*

$$\mathfrak{n}_0(\mathbf{1}_M) = 0.$$

Proof. The proof is based on the monotonicity and proceed as follows. We decompose $R(M)$ to the connected components $R(M) = \bigcup_{i\in I} R_i$. We first observe

$$\mathcal{M}((M,\mathcal{E}), L; R_i, R_j; E) = \emptyset$$

for $i \neq j$. In fact the boundary value of an element of $\mathcal{M}((M,\mathcal{E}), L; R_i, R_j; E)$ is a path joining R_i and R_j in $R(M)$. We next show:

Lemma 3.11. *If $E > 0$ then*

$$\dim \mathcal{M}((M,\mathcal{E}), L; R_i, R_i; E) > \dim \mathcal{M}((M,\mathcal{E}), L; R_i, R_i; 0).$$

Here dim *means the virtual dimension.*

Proof. Let $(\mathfrak{A}, \Omega, u)$ be an representative of an element of $\mathcal{M}((M,\mathcal{E}), L; R_i, R_i; E)$. The boundary value of u defines a $t \in \mathbb{R}$ parametrized family $a(1,t)$ of flat connections of M. We may regard it as a connection on $M \times \mathbb{R}$, which we denote by $\mathfrak{A}_0$. We remark that $\mathfrak{A}$ and $\mathfrak{A}_0$ has the same boundary value on $\Sigma \times \{1\} \times \mathbb{R}$ and the same asymptotic as $\mathbb{R}$ coordinate goes to $\pm\infty$. So we can define relative Pontryagin number

$$\int_{M\times\mathbb{R}} (p_1(\mathfrak{A}) - p_1(\mathfrak{A}_0)) \in \mathbb{Z}.$$

We then have

$$2\pi^2 \int_{M\times\mathbb{R}} (p_1(\mathfrak{A}) - p_1(\mathfrak{A}_0)) = E.$$

Since E is strictly positive $\int_{M\times\mathbb{R}} (p_1(\mathfrak{A}) - p_1(\mathfrak{A}_0))$ is strictly positive. Therefore $\mathfrak{A}$ is homotopic relative to the boundary to a connection obtained by gluing $\mathfrak{A}_0$ with a connection on S^4 with positive Pontryagin number. Lemma 3.11 now follows from index sum formula. □

Proposition 3.10 follows from Lemma 3.11 and dimension counting. □

Proposition 3.10 implies that $b_M = 0$ if $R(M)$ is an embedded Lagrangian submanifold. This is Statement (3) of Theorem 1.1.

Remark 3.12. We can use Proposition 3.5 to study Wehrheim-Woodward functoriality ([WW1]) in a similar way. Especially we can prove the next theorem. Let (M_i, ω_i) $i = 1, 2$ be a symplectic manifold, $i_{L_1} : \tilde L_1 \to L_1 \subset M_1$ an immersed Lagrangian submanifold of M_1 and $i_{L_{12}} : \tilde L_{12} \to L_{12} \subset M_1 \times M_2$ an immersed Lagrangian submanifold of $(M_1 \times M_2, \omega_1 \oplus -\omega_2)$. We assume $L_1 \times M_2$ is transversal to L_{12} and put

$$\tilde L_2 = (L_1 \times M_2) \times_{M_1 \times M_2} \tilde L_{12}.$$

Then the composition of $\tilde L_2 \subset M_1 \times M_2$ with the projection $M_1 \times M_2 \to M_2$ is a Lagrangian immersion $i_{L_2} : \tilde L_2 \to L_2 \subset M_2$. We assume M_1, M_2, L_1, L_{12} are spin and fix a spin structure of them. It induces a spin structure of L_2.

Theorem 3.13. *In the above situation we assume that L_1 and L_{12} are unobstructed, in addition.*

Then L_2 is unobstructed. Moreover gauge equivalence classes of the bounding cochains b_1, b_{12} of L_1 and L_{12} determine a gauge equivalence class of a bounding cochain b_2 of L_2.

For the proof we replace Figure 2.9 by the next Figure 3.1.

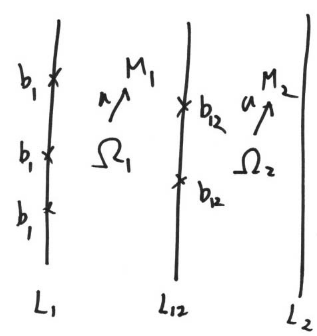

Figure 3.1

Here u is a combination of maps, to M_1 (in the domain Ω_1) and M_2 (in the domain Ω_2). L_1 and L_{12} are used to define a boundary condition on $\partial\Omega_1 \setminus (\Omega_1 \cap \Omega_2)$ and on $\Omega_1 \cap \Omega_2$, respectively. We use b_1, the bounding cochain of L_1 and b_{12}, the bounding cochain of L_{12} to cancel the (disk) bubbles on $\partial\Omega_1 \setminus (\Omega_1 \cap \Omega_2)$ and on $\Omega_1 \cap \Omega_2$, respectively. We thus obtain a structure of filtered A_∞ right module over $CF(L_2)$. Using the fact that $\tilde L_2 \times_{M_2} \tilde L_2 = \tilde L_1 \times_{M_1} \tilde L_{12} \times_{M_2} \tilde L_2$, we can show $[\tilde L_2]$ is the cyclic element of this filtered A_∞ right module and can apply Proposition 3.5.

We will prove Theorem 3.13 and explore related topics in [Fu9]. See also Remark 4.15.

4. Representativity of the relative Floer homology functor

In this section we explain a proof of Theorem 1.4. Let $(M, \mathcal{E}_M)$ and $(\Sigma, \mathcal{E}_\Sigma)$ be as in Situation 2.1. In Theorem 3.9 we obtain a bounding cochain b_M of the filtered A_∞ algebra $(CF(R(M)), \{\mathfrak{m}_k\})$ satisfying (3.9).

We consider a Lagrangian immersion $i_L : \tilde L \to R(\Sigma)$ and its filtered A_∞ algebra $(CF(L), \{\mathfrak{m}_k\})$ where $CF(L) = C(\tilde L \times_{R(\Sigma)} \tilde L; \Lambda_0^{\mathbb{Z}_2})$. In Section 2 we defined a right filtered A_∞ module $(CF(M, L), \{\mathfrak{n}_k\})$ over $(CF(L), \{\mathfrak{m}_k\})$ where $CF(M, L) = C(R(M) \times_{R(\Sigma)} \tilde L; \Lambda_0^{\mathbb{Z}_2})$.

We define right filtered A_∞ module $(CF((R(M),b_M),L),\{{}^{b_M}\mathfrak{n}_k\})$ over $(CF(L),\{\mathfrak{m}_k\})$ as follows. We put

$$ {}^{b_M}\mathfrak{n}_k(y;x_1,\dots,x_k) = \sum_{\ell=0}^{\infty}\mathfrak{n}_{\ell,k}(\underbrace{b_M,\dots,b_M}_{\ell};y;x_1,\dots,x_k). \tag{4.1}$$

Here $\{\mathfrak{n}_{\ell,k}\}$ is the filtered A_∞ bimodule structure of $CF(R(M),L)$ over $CF(R(M))$-$CF(L)$. We review the construction of this filtered A_∞ bimodule structure later. See Corollary 4.9.

Using the fact that b_M satisfies the Maurer-Cartin equation (3.1) we can prove that ${}^{b_M}\mathfrak{n}_k$ defines the structure of right filtered A_∞ module. Namely it satisfies (2.7).

We recall:

Definition 4.1. Let $(D_i,\{\mathfrak{n}^i_k\})$ $(i=1,2)$ be right filtered A_∞ modules over $(C,\{\mathfrak{m}_k\})$. A *filtered A_∞ homomorphism* : $(D_1,\{\mathfrak{n}^1_k\})\to(D_2,\{\mathfrak{n}^2_k\})$ is $\widehat{\varphi}=\{\varphi_k\mid k=0,1,2,\dots\}$ where

$$\varphi_k : D_1\otimes C^{k\otimes}\to D_2$$

such that

$$\begin{aligned}&\sum\mathfrak{n}_{k_1}(\varphi_{k_2}(y;x_1,\dots,x_{k_2});\dots,x_k)\\ =&\sum\varphi_{k_1}(\mathfrak{n}_{k_2}(y;x_1,\dots,x_{k_2});\dots,x_k)\\ &+\sum\varphi_{k_1}(y;x_1,\dots,\mathfrak{m}_{k_2}(x_i,\dots,x_{i+k_2-1}),\dots,x_k).\end{aligned}\tag{4.2}$$

Here the sums in the first and the second lines are taken over k_1,k_2 with $k_1+k_2=k$, the sum in the third line is taken over k_1,k_2,i such that $k_1+k_2=k+1$ and $i=1,\dots,k_1$.

We say $\widehat{\varphi}$ is *strict* if $\varphi_0=0$.

We will prove:

Theorem 4.2. *There exists a strict right filtered A_∞ module homomorphism*

$$\hat{\varphi} : (CF((R(M),b_M),L),\{{}^{b_M}\mathfrak{n}_k\})\to(CF(M;L),\{\mathfrak{n}_k\})) \tag{4.3}$$

over $(CF(L),\{\mathfrak{m}_k\})$ such that

$$\varphi_1\equiv \mathrm{id}\mod \Lambda_+^{\mathbb{Z}_2}. \tag{4.4}$$

Here $\hat{\varphi}=\{\varphi_k\mid k=1,2,\dots\}$.

We remark that $CF((R(M),b_M),L)$ and $CF(M,L)$ are both $C(R(M)\times_{R(\Sigma)} L;\Lambda_0^{\mathbb{Z}_2})$ as $\Lambda_0^{\mathbb{Z}_2}$ modules. So the identity map in the right hand side of (4.4) makes sense.

Before proving Theorem 4.2 we draw its consequence.

Let b be a bounding cochain of $(C(L),\{\mathfrak{m}_k\})$ we define $d^b : C(M,L)\to C(M;L)$ by (2.8). Then $d^b\circ d^b=0$ and the Floer cohomology $HF(M,(L,b))$ is the cohomlogy group of d^b.

In the same way we define $d^b=({}^{b_M}\mathfrak{n}^b)_0 : CF((R(M),b_M),L)\to CF((R(M),b_M),L)$ by

$$d^b(y)=\sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{m}_{\ell+k+1}(\underbrace{b_M,\dots,b_M}_{\ell},y,\underbrace{b,\dots,b}_{k}).$$

$d^b\circ d^b=0$ again holds and $HF((R(M),b_M),(L,b))$ is its cohomology group.

Corollary 4.3. *There exists a canonical isomorphism*

$$HF((R(M), b_M), (L, b)) \cong HF(M, (L, b)). \tag{4.5}$$

Proof of Theorem 4.2 $\Rightarrow$ Corollary 4.3. We define a map

$$\varphi^b : CF((R(M), b_M), L) \to CF(M, L)$$

by

$$\varphi^b(y) = \sum_{k=0}^{\infty} \varphi_k(y; b, \dots, b).$$

Since $b \equiv 0 \mod \Lambda_+^{\mathbb{Z}_2}$ the right hand side converges in T-adic topology. It is easy to check that (4.2) and (3.1) imply $d^b \circ \varphi^b = \varphi^b \circ d^b$. Namely φ^b is a chain map. Then (4.4) and $b \equiv 0 \mod \Lambda_+^{\mathbb{Z}_2}$ implies that $\varphi^b \equiv \mathrm{id} \mod \Lambda_+^{\mathbb{Z}_2}$. Therefore φ^b induces an isomorphism on cohomologies. □

Proof of Theorem 4.2. The main part of the proof is defining the moduli spaces which we use to define the operators φ_k.

We take a domain $W \subset \mathbb{C}$ such that the following holds. (See Figure 4.1.)

Condition 4.4.
(1) The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im} z < -2\}$ is $\{z \in \mathbb{C} \mid |\mathrm{Re} z| \le 1, \mathrm{Im} z < -2\}$.
(2) The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im} z > +2\}$ is $\{z \in \mathbb{C} \mid |\mathrm{Re} z| \le 1, \mathrm{Im} z > +2\}$.
(3) The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Re} z > 0\}$ is $\{z \in \mathbb{C} \mid 0 < \mathrm{Re} z \le 1\}$.
(4) The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im} z < -2\}$ is $\{z \in \mathbb{C} \mid \mathrm{Im} z < -2, |\mathrm{Re} z| \le 1\}$.
(5) The boundary ∂W has three connected components $\partial_i W$ ($i = 1, 2, 3$) each of which is a C^∞ submanifold of $\mathbb{C}$ and is diffeomorphic to $\mathbb{R}$. Moreover $\partial_1 W = \{z \in \mathbb{C} \mid \mathrm{Re} z = 1\}$, $\partial_2 W \subset \{z \in \mathbb{C} \mid \mathrm{Re} z < 0, \mathrm{Im} z > 0\}$, $\partial_3 W \subset \{z \in \mathbb{C} \mid \mathrm{Re} z < 0, \mathrm{Im} z < 0\}$.

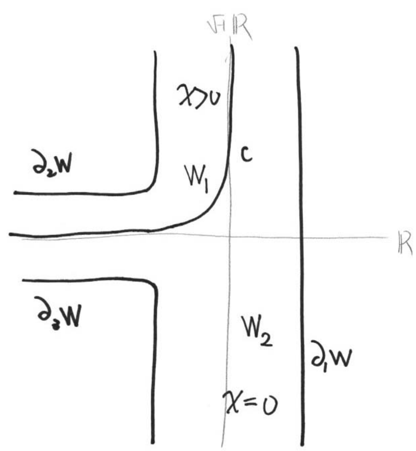

Figure 4.1

We take $\chi : W \to [0, 1]$, a submanifold $C \subset W$ diffeomorphic to $\mathbb{R}$, and a Riemannian metric $\mathbf{g}$ on $\Sigma \times W_1$ with the following properties: (See Figure 4.1.)

Condition 4.5.
(1) $W \setminus C$ consists of two connected components W_1 and W_2. Moreover $\{z \in \mathbb{C} \mid \chi(z) > 0\} = W_1$.

(2) On $\{z \in W \mid \mathrm{Re}z < -3\}$, $\chi(z) = \chi(-\mathrm{Im}z)$, where χ in the right hand side is the same function as one appeared in (2.11). $\mathbf{g} = \chi^2 g_\Sigma + ds^2 + dt^2$ on $\Sigma \times \{z \in W \mid \mathrm{Re}z < -3\}$, where we put $z = t - \sqrt{-1}s$.

(3) On $\{z \in W \mid \mathrm{Im}z < -3\}$, $\chi(z) = \chi(\mathrm{Re}z)$, where χ in the right hand side is the same function as one appeared in (2.11). $\mathbf{g} = \chi^2 g_\Sigma + ds^2 + dt^2$ on $\Sigma \times \{z \in W \mid \mathrm{Im}z < -3\}$, where we put $z = s + \sqrt{-1}t$.

(4) In a neighborhood of $\Sigma \times \partial_2 W$, the space $\Sigma \times W_1$ with metric $\mathbf{g}$ is isometric to the direct product $g_\Sigma \times (0, \epsilon) \times \mathbb{R}$. Here $g_\Sigma \times \{0\} \times \mathbb{R}$ corresponds to $\Sigma \times \partial_2 W$. This isometry is compatible with the isometry obtained by items (2)(3) in the domain described by those items.

(5) Let $U(C)$ be a neighborhood of C in $\mathbb{C}$. Then on $\Sigma \times (W_1 \cap U(C))$, the metric $\mathbf{g}$ becomes $\chi^2(s,t) g_\Sigma + ds^2 + dt^2$. where $s + \sqrt{-1}t$ is the standard coordinate of $\mathbb{C}$ and χ satisfies the condition of [Fu5, Lemma 4.7].

We extend the metric $\mathbf{g}$ on $\Sigma \times W_1$ to a 'singular metric' on $\Sigma \times W$ by putting $\mathbf{g} = 0 g_\Sigma + ds^2 + dt^2$ outside $\Sigma \times W_1$.

By Condition 4.5 (4), $(\Sigma \times W_1, \mathbf{g})$ is isometric to $(\Sigma \times (0, \epsilon) \times \mathbb{R}, g_\Sigma + ds^2 + dt^2)$ in a neighborhood of $\Sigma \times \partial_2 W$.

We remark that $M_0 \times \mathbb{R}$ is isometric to $(\Sigma \times (-\epsilon, 0) \times \mathbb{R}, g_\Sigma + ds^2 + dt^2)$ in an neighborhood of its boundary. Therefore we can glue them together to obtain $(X, \mathbf{g})$. Here $\mathbf{g}$ is a 'Riemannian metric' which is degenerate on $\Sigma \times \overline{W_2}$. The $SO(3)$ bundle $\mathcal{E}_M$ on M induces an $SO(3)$ bundle on X in an obvious way, which we denote by $\mathcal{E}_X$.

Note that X has 3 ends and 2 boundary components. The 3 ends appear as $\mathrm{Im}z \to +\infty$, $\mathrm{Re}z \to -\infty$, $\mathrm{Im}z \to -\infty$.

The end corresponding to $\mathrm{Im}z \to +\infty$ is $M \times [c, \infty)$. The end corresponding to $\mathrm{Re}z \to -\infty$ is $M \times (-\infty, -c]$. The end corresponding to $\mathrm{Im}z \to -\infty$ is $\Sigma \times [-1, 1] \times (-\infty, -c]$.

The boundaries are $\Sigma \times \partial_1 W$ and $\Sigma \times \partial_3 W$. Note $\Sigma \times \partial_2 W$ is glued with $\partial M_0 \times \mathbb{R}$ and so is not a boundary of X.

For a smooth connection $\mathfrak{A}$ of $\mathcal{E}_X$ we can consider the 'ASD-equation'. Namely we require (2.12) on $X \setminus (\Sigma \times \overline{W_2})$ and (2.14) on $\Sigma \times W \subset X$. (Note (2.12) coincides with (2.14) on the overlapped part.) We say $\mathfrak{A}$ is an ASD-connection by an abuse of notation if it satisfies (2.12) on $X \setminus (\Sigma \times \overline{W_2})$ and (2.14) on $\Sigma \times W \subset X$.

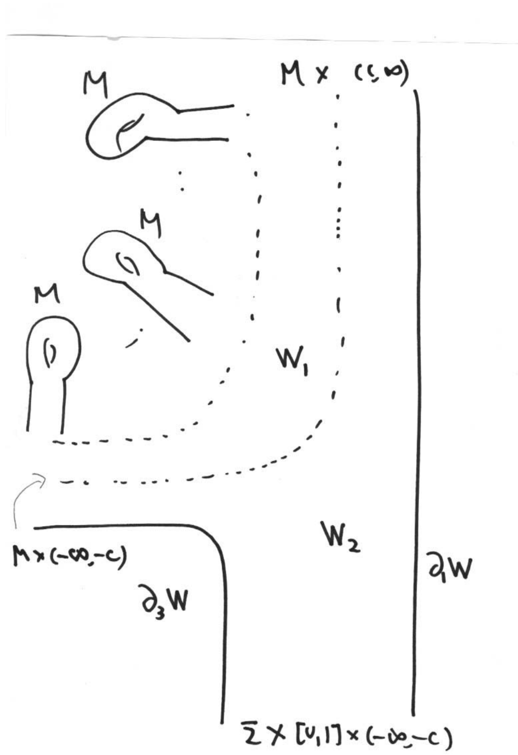

Figure 4.2

We consider also Ω which contains W_2. Namely Ω is a union of W_2 and tree of disk and sphere components attached to ∂W_2 and $\mathrm{Int}W_2$, respectively. We consider the pair (Ω, u) which satisfies Condition 2.7, except we replace $(0,1)\times\mathbb{R}$ and $\{0\}\times\mathbb{R}$ by W_2 and C, respectively. We call this condition Condition 2.7'.

Let $\partial_1\Omega$ (resp. $\partial_3\Omega$) be the union of $\partial_1 W$ (resp. $\partial_3 W$) and the boundary of the disk components attached to $\partial_1 W$ (resp. $\partial_3 W$).

Now we modify Definitions 2.9 and 2.23 as follows. We consider the decompositions (2.37) and (2.36). Let $I(L)$ and $I(R(M),L)$ be the index sets as in there. We consider the decomposition (2.36) in case $L = R(M)$ and let $I(R(M))$ be its index set. Namely

$$\tilde L \times_{R(\Sigma)} \tilde L = \bigcup_{j\in I(L)} \hat L_j$$

$$R(M) \times_{R(\Sigma)} R(M) = \bigcup_{j\in I(R(M))} \widehat{R(M)}_j$$

$$R(M) \times_{R(\Sigma)} \tilde L = \bigcup_{j\in I(R(M),L)} R_i.$$

Let R_1 be a connected component of $R(M)\times_{R(\Sigma)} R(M)$ and R_2, R_3 be connected components of $R(M)\times_{R(\Sigma)}\tilde L$. In other words $R_1 = \widehat{R(M)}_{j_1}$ for some $j_1 \in I(R(M))$. $R_2 = R_{j_2}$, $R_3 = R_{j_3}$ for some $j_2, j_3 \in I(R(M),L)$.

We also take $\vec{i}^{(1)}$, $\vec{i}^{(3)}$ as in Definition 4.6 (5) below.

Definition 4.6. We define the set $\overset{\circ}{\mathcal{M}}((X,\mathcal{E}_X),R(M),L;R_1,R_2,R_3;\vec{i}^{(1)},\vec{i}^{(3)};E)$ as the set of all equivalence classes of $(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z}^{(1)},\vec{z}^{(3)})$ satisfying the following conditions. (See Figure 4.3.)

(1) $\mathfrak{A}$ is a connection of $\mathcal{E}_X$ satisfying equations (2.12), (2.14).

(2) $\mathfrak{z}=(\mathfrak{z}_1,\dots,\mathfrak{z}_{m_1})$ is an *unordered* m_1-tuple of points of $X\setminus(\Sigma\times\overline{W_2})$. We put $\|\mathfrak{z}\|=m_1$. We say subset $\{\mathfrak{z}_1,\dots,\mathfrak{z}_{m_1}\}\subset X\setminus(\Sigma\times\overline{W_2})$ the *support* of $\mathfrak{z}$ and denote it by $|\mathfrak{z}|$. We define multi : $|\mathfrak{z}|\to\mathbb{Z}_{>0}$ by $\mathrm{multi}(x)=\#\{i\mid z_i=x\}$ and call it the *multiplicity function*.

(3) $\mathfrak{w}=(\mathfrak{w}_1,\dots,\mathfrak{w}_{m_2})$ is an *unordered* m_2-tuple of points of C. We put $\|\mathfrak{w}\|=m_2$. We say the subset $\{\mathfrak{w}_1,\dots,\mathfrak{w}_{m_2}\}\subset C$ the *support* of $\mathfrak{w}$. We define multi : $|\mathfrak{w}|\to\mathbb{Z}_{>0}$ by $\mathrm{multi}(x)=\#\{i\mid w_i=x\}$ and call it the *multiplicity function*.

(4) Ω satisfies Condition 2.7'.

(5) $\vec{i}^{(1)}=(i^{(1)}(1),\dots,i^{(1)}(k_1))\in I(L)^{k_1}$ and $\vec{i}^{(3)}=(i^{(3)}(1),\dots,i^{(3)}(k_3))\in I(R(M))^{k_3}$

(6) $\vec{z}^{(1)}=(z^{(1)}_1,\dots,z^{(1)}_{k_1})$ (resp. $\vec{z}^{(3)}=(z^{(3)}_1,\dots,z^{(3)}_{k_3})$) $z^{(1)}_i$ lies on $\partial_1\Omega$, (resp. $z^{(3)}_i$ lies on $\partial_3\Omega$). None of $z^{(1)}_i$ or $z^{(3)}_i$ is a nodal point. $z^{(1)}_i\neq z^{(1)}_j$, $z^{(3)}_i\neq z^{(3)}_j$ if $i\neq j$. $(z^{(1)}_1,\dots,z^{(1)}_{k_1})$ (resp. $(z^{(3)}_1,\dots,z^{(3)}_{k_3})$) respects counter clockwise orientation of $\partial_1\Omega$ (resp. $\partial_3\Omega$).

(7) There exists $\gamma^{(1)}:\partial_1\Omega\setminus\{z^{(1)}_1,\dots,z^{(1)}_{k_1}\}\to\tilde{L}$ such that $u(z)=i_L(\gamma^{(1)}(z))$ on $\partial_1\Omega\setminus\{z^{(1)}_1,\dots,z^{(1)}_{k_1}\}$.

There exists $\gamma^{(3)}:\partial_3\Omega\setminus\{z^{(3)}_1,\dots,z^{(3)}_{k_3}\}\to R(M)$ such that $u(z)=i_{R(M)}(\gamma^{(3)}(z))$ on $\partial_3\Omega\setminus\{z^{(3)}_1,\dots,z^{(3)}_{k_3}\}$.

(8) For $j=1,\dots,k_1$ the following holds.

$$(\lim_{z\uparrow z^{(1)}_j}\gamma^{(1)}(z),\lim_{z\downarrow z^{(1)}_j}\gamma^{(1)}(z))\in\hat{L}_{i^{(1)}(j)}. \tag{4.6}$$

Here the notation $z\uparrow z_j$, $z\downarrow z_j$ is defined in the same way as (2.38)

For $j=1,\dots,k_3$ the following holds.

$$(\lim_{z\uparrow z^{(3)}_j}\gamma^{(3)}(z),\lim_{z\downarrow z^{(3)}_j}\gamma^{(3)}(z))\in\widehat{R(M)}_{i^{(3)}(j)}. \tag{4.7}$$

Here the notation $z\uparrow z_j$, $z\downarrow z_j$ is defined in the same way as (2.38)[5].

(9) We replace Condition 2.8 (3) by the stability of $(\Omega,u,\vec{z}^{(1)},\vec{z}^{(3)})$. Namely the set of all maps $v:\Omega\to\Omega$ satisfying the next three conditions is a finite set.
  (a) v is a homeomorphism and is holomorphic on each of the irreducible components.
  (b) v is the identity map on $(0,1]\times\mathbb{R}\subseteq\Omega$.
  (c) $u\circ v=u$.
  (d) $v(z^{(1)}_j)=z^{(1)}_j$, $j=1,\dots,k_1$ and $v(z^{(3)}_j)=z^{(3)}_j$, $j=1,\dots,k_3$.

(10) For $(s,t)\in W_2$ we have

$$[A(s,t)]=u(s,t).$$

[5]We remark that $z\uparrow z_j$ here means that z moves to the counter-clock-wise way towards z. So $\mathrm{Im}z>\mathrm{Im}z_j$.

Here $A(s,t)$ is obtained from $\mathfrak{A}$ by (2.13).

(11) The energy of $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ which is defined in the same way as Definition 2.10 is E.

(12) We assume the following asymptotic boundary conditions, which are defined by using R_1, R_2, R_3.

(a)
$$(\lim_{z \to -1-\infty\sqrt{-1}} \gamma^{(3)}(z), \lim_{z \to +1-\infty\sqrt{-1}} \gamma^{(1)}(z)) \in R_1. \tag{4.8}$$
Here $\lim_{z \to -1-\infty\sqrt{-1}}$ is the limit when the imaginary part of $z \in \partial_3 W$ goes to $-\infty$. (We remark that then the $\mathrm{Re} z = -1$ by Condition 4.4 (4)(5).) The meaning of $\lim_{z \to +1-\infty\sqrt{-1}}$ is similar.

(b) We consider the restriction of $\mathfrak{A}$ to $\Sigma \times \{z \in W \mid \mathrm{Im} z = c\}$ for $c > 3$. We glue it with the restriction of $\mathfrak{A}$ to $M_0 = M \setminus (\Sigma \times (-1, 1])$, which is attached to $-1 + c\sqrt{-1}$. (See Figure 4.4.) We call it $\mathfrak{A}|_{\mathrm{Im} z = c}$. It is a connection of the bundle $\mathcal{E}_M$ on M. We assume that $\mathfrak{A}|_{\mathrm{Im} z = c}$ converges to a flat connection as $c \to +\infty$. We write its limit $\lim_{c \to +\infty} \mathfrak{A}|_{\mathrm{Im} z = c}$. Then we also assume
$$(\lim_{c \to +\infty} \mathfrak{A}|_{\mathrm{Im} z = c}, \lim_{z \to +1+\infty\sqrt{-1}} \gamma^{(1)}(z)) \in R_2. \tag{4.9}$$
Here the meaning of $\lim_{z \to +1+\infty\sqrt{-1}}$ is similar to (4.8).

(c) We consider the restriction of $\mathfrak{A}$ to $\Sigma \times \{z \in W \mid \mathrm{Re} z = c\}$ for $c < -3$. We glue it with the restriction $\mathfrak{A}$ to M_0 which is attached to $c + \sqrt{-1}$. (See Figure 4.5.) We call it $\mathfrak{A}|_{\mathrm{Re} z = c}$. It is a connection of the bundle $\mathcal{E}_M$ on M. We assume that $\mathfrak{A}|_{\mathrm{Re} z = c}$ converges to a flat connection as $c \to -\infty$. We write its limit $\lim_{c \to -\infty} \mathfrak{A}|_{\mathrm{Re} z = c}$.
Then we also assume
$$(\lim_{c \to -\infty} \mathfrak{A}|_{\mathrm{Re} z = c}, \lim_{z \to -\infty-\sqrt{-1}} \gamma^{(3)}(z)) \in R_3. \tag{4.10}$$
Here the meaning of $\lim_{z \to +1+\infty\sqrt{-1}}$ is similar to (4.8).

The equivalence relation is defined in the same way as Definition 2.11.

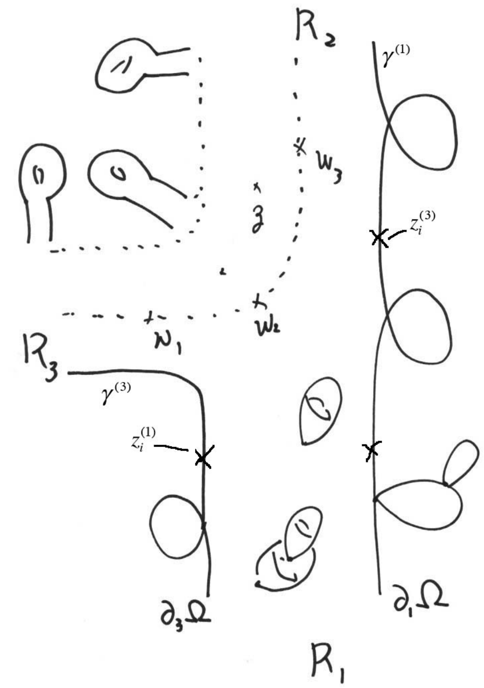

Figure 4.3

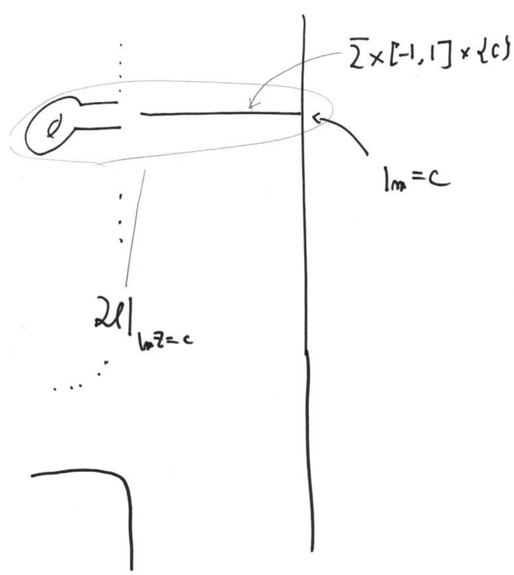

Figure 4.4

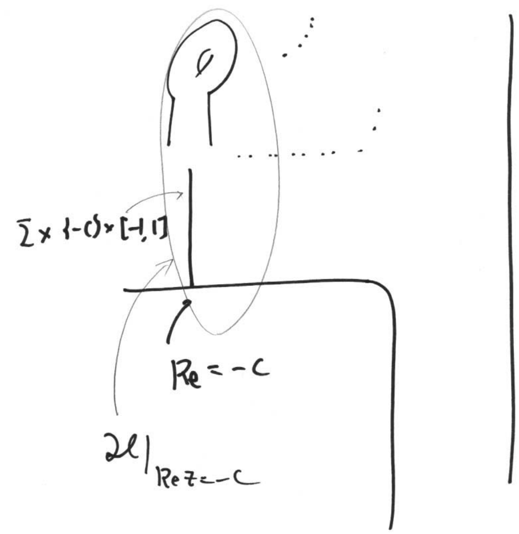

Figure 4.5

We can define a topology on $\overset{\circ}{\mathcal{M}}((X,\mathcal{E}_X),R(M),L;R_1,R_2,R_3;\vec{i}^{(1)},\vec{i}^{(3)};E)$ by modifying Definition 2.12 in an obvious way.

We put

$$\begin{aligned}&\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)\\&=\bigcup_{\vec{i}^{(1)};|\vec{i}^{(1)}|=k_1}\ \bigcup_{\vec{i}^{(3)};|\vec{i}^{(3)}|=k_3}\widetilde{\overset{\circ}{\mathcal{M}}}((X,\mathcal{E}_X),L;R_1,R_2,R_3;\vec{i}^{(1)},\vec{i}^{(3)};E)\end{aligned}$$

$\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)$ is a Hausdorff space.

We define evaluation maps

$$\begin{aligned}&\mathrm{ev}^{(1)}:\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)\to(\tilde{L}\times_{R(\Sigma)}\tilde{L})^{k_1}\\&\mathrm{ev}^{(3)}:\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)\to(R(M)\times_{R(\Sigma)}R(M))^{k_3}\end{aligned}\tag{4.11}$$

They are defined by (4.6) and (4.7).

We also define the evaluation maps

$$\mathrm{ev}_i^{\infty}:\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)\to R_i\tag{4.12}$$

for $i=1,2,3$. They are defined by (4.8), (4.9) and (4.10).

Note $\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)$ is not yet compact. There are still three types of ends, which are:

(I) An ASD-connection escape to the direction $\mathrm{Im}(z)\to+\infty$.
(II) An ASD-connection escape to the direction $\mathrm{Re}(z)\to-\infty$.
(III) A pseudo-holomorphic strip escape to the direction $\mathrm{Im}(z)\to-\infty$.

The end (I) is described by the union of the fiber products:

$$\overset{\circ}{\mathcal{M}}_{k_1',k_3}((X,\mathcal{E}_X),L;R_1,R_2',R_3;E_1)_{\mathrm{ev}_2^{\infty}}\times_{\mathrm{ev}_-}\mathcal{M}_{k_1''}((M,\mathcal{E}),L;R_2',R_2;E_2)\tag{4.13}$$

Here the union is taken over all k'_1, k''_1, E_1, E_2, R'_2 such that $k'_1+k''_1 = k_1$, $E_1+E_2 = E$ and R'_2 is a connected component of $R(M)\times_{R(\Sigma)}\tilde{L}$. Note $\mathcal{M}_{k''_1}((M,\mathcal{E}),L;R'_2,R_2;E_2)$ is defined in Definition 2.23. See Figure 4.6.

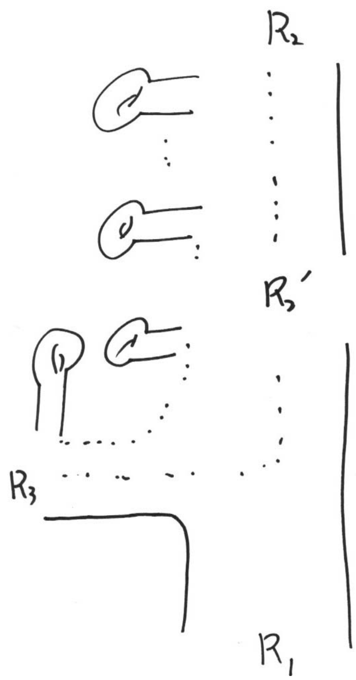

Figure 4.6

The end (II) is described by the union of the fiber products:

$$\begin{aligned}&\overset{\circ}{\mathcal{M}}_{k_1,k'_3}((X,\mathcal{E}_X),L;R_1,R_2,R'_3;E_1)\\ &{}_{\mathrm{ev}_3^{\infty}}\times_{\mathrm{ev}_+}\mathcal{M}_{k''_3}((M,\mathcal{E}),R(M);R_3,R'_3;E_2)\end{aligned} \tag{4.14}$$

Here the union is taken over all k'_3, k''_3, E_1, E_2, R'_3 such that $k'_3 + k''_3 = k_3$, $E_1 + E_2 = E$ and R'_3 is a connected component of $R(M) \times_{R(\Sigma)} R(M)$. See Figure 4.7. We remark that in the second line R_3 appears first and R'_3 next. (Namely R_3, R'_3 and not R'_3, R_3.) The reason is as follows. In Figure 4.7 'the bubble' component lies in the left. We need to rotate it by 90 degree counter-clock-wise direction to put it in the same way as Figure 2.2. Then after rotation, R_3 will lie in the direction $\mathrm{Im} z \to -\infty$.

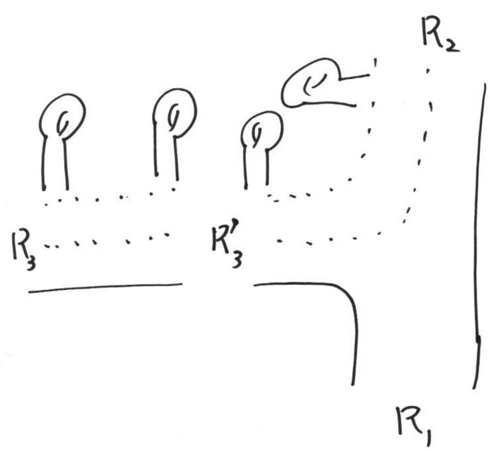

Figure 4.7

To describe the end (III) we use the moduli space of pseudo-holomorphic strips which is used to define Lagrangian Floer homology $HF(R(M), L)$. We here need a digression and review the definition of $HF(R(M), L)$. The discussion below is a generalization of [FOOO1, Section 3.7.4] to the case of a pair of *immersed* Lagrangian submanifolds. [AJ] did not discuss the case of pairs of immersed Lagrangian submanifolds since we may regard the union as a single immersed Lagrangian submanifold and so it is actually included in the case of single immersed Lagrangian submanifold. Therefore the only point which is not literally written in [AJ] is that we include the case when the intersection of two immersed Lagrangian submanifolds are clean but not transversal. (This generalization is not a big deal and one can handle it in the same way as [FOOO1, Section 3.7.4].)

It seems more natural to explain it in a general situation rather than a special case we use here. Let (Y, ω) be a monotone symplectic manifold, (We assume monotonicity here since we use $\mathbb{Z}_2$ coefficient.) and $i_{L_j} : \tilde L_j \to Y$ be Lagrangian immersion for $j = 1, 2$. We fix a compatible almost complex structure J_Y on Y. We assume that the self-intersection of L_j are transversal for $j = 1, 2$ and the fiber product $\tilde L_1 \times_Y \tilde L_2$ is clean. We decompose the fiber products into connected components and put:

$$\tilde L_j \times_Y \tilde L_j = \bigcup_{k \in I(L_j)} \hat L_{j,k} \quad j = 1, 2,$$
$$\tilde L_1 \times_Y \tilde L_2 = \bigcup_{k \in I(L_1, L_2))} \widehat R_k$$

Let R_-, R_+ be connected components of $\tilde L_1 \times_Y \tilde L_2$. Let $\vec i^{(j)} = (i^{(j)}(1), \dots, i^{(j)}(k_j))$ where $i^{(j)}(1), \dots, i^{(j)}(k_j) \in I(L_j)$.

We define the moduli space $\widetilde{\overset{\circ}{\mathcal{M}}}(L_1, L_2; R_-, R_+; \vec i^{(1)}, \vec i^{(2)}; E)$ as follows.

Definition 4.7. The moduli space $\overset{\circ}{\mathcal{M}}(L_1, L_2; R_-, R_+; \vec i^{(1)}, \vec i^{(2)}; E)$ is the set of all the equivalence classes of $(\Omega, u, \vec z^{(1)}, \vec z^{(2)})$ with the following properties. (See Figure 4.8.)

(1) Ω is a union of the domain

$$\Omega_0 = \{z \in \mathbb{C} \mid |\mathrm{Re} z| \le 1\}, \tag{4.15}$$

trees of spheres attached to the interior of Ω_0 and trees of disk components attached to the boundary of Ω_0. (The disk components may contain a tree of sphere components attached to its interior.) We denote by $\partial_1\Omega$ (resp. $\partial_2\Omega$) the union of $\{z \in \Omega \mid \mathrm{Rez} = -1\}$ and the boundaries of the tree of disks attached to it (resp. the union of $\{z \in \Omega \mid \mathrm{Rez} = 1\}$ and the boundaries of the tree of disks attached to it.)

We remark $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$.

(2) $u : \Omega \to R(\Sigma)$ is a pseudo-holomorphic map.

(3) $\vec{z}^{(1)} = (z_1^{(1)}, \dots, z_{k_1}^{(1)})$ (resp. $\vec{z}^{(2)} = (z_1^{(2)}, \dots, z_{k_2}^{(2)})$) $z_i^{(1)}$ lies on $\partial_1\Omega$, (resp. $z_i^{(2)}$ lies on $\partial_2\Omega$). None of $z_i^{(1)}$ or $z_i^{(2)}$ is a nodal point. $z_i^{(1)} \ne z_j^{(1)}$, $z_i^{(2)} \ne z_j^{(2)}$ if $i \ne j$. $(z_1^{(1)}, \dots, z_{k_1}^{(1)})$ $(z_1^{(1)}, \dots, z_{k_1}^{(1)})$ (resp. $(z_1^{(2)}, \dots, z_{k_2}^{(2)})$) respects counter clockwise orientation of $\partial_1\Omega$ (resp. $\partial_2\Omega$).

(4) There exists $\gamma^{(1)} : \partial_1\Omega \setminus \{z_1^{(1)}, \dots, z_{k_1}^{(1)}\} \to \tilde{L}_1$ such that $u(z) = i_{L_1}(\gamma(z))$ on $\partial_1\Omega \setminus \{z_1^{(1)}, \dots, z_{k_1}^{(1)}\}$.

There exists $\gamma^{(2)} : \partial_2\Omega \setminus \{z_1^{(2)}, \dots, z_{k_2}^{(2)}\} \to \tilde{L}_2$ such that $u(z) = i_{L_2}(\gamma(z))$ on $\partial_2\Omega \setminus \{z_1^{(2)}, \dots, z_{k_2}^{(2)}\}$.

(5) For $j = 1, \dots, k_1$ the following holds.

$$\left(\lim_{z \uparrow z_j^{(1)}} \gamma^{(1)}(z), \lim_{z \downarrow z_j^{(1)}} \gamma^{(1)}(z)\right) \in \hat{L}_{1,i(j)}. \tag{4.16}$$

Here the notation $z \uparrow z_j$, $z \downarrow z_j$ is defined in the same way as (2.38)

For $j = 1, \dots, k_2$ the following holds.

$$\left(\lim_{z \uparrow z_j^{(2)}} \gamma^{(2)}(z), \lim_{z \downarrow z_j^{(2)}} \gamma^{(2)}(z)\right) \in \hat{L}_{2,i(j)}. \tag{4.17}$$

Here the notation $z \uparrow z_j$, $z \downarrow z_j$ is defined in the same way as (2.38).

(6) We asuume the stability of $(\Omega, u, \vec{z}^{(1)}, \vec{z}^{(2)})$. Namely the set of all maps $v : \Omega \to \Omega$ satisfying the next three conditions is a finite set.
   (a) v is a homeomorphism and is holomorphic on each of the irreducible components.
   (b) v is the identity map on Ω_0.
   (c) $u \circ v = u$.
   (d) $v(z_j^{(1)}) = z_j^{(1)}$, $j = 1, \dots, k_1$ and $v(z_j^{(2)}) = z_j^{(2)}$, $j = 1, \dots, k_2$.

(7) The energy of u is E. Namely

$$\int_\Omega u^*\omega = E.$$

(8) We assume the following asymptotic boundary conditions, which are defined by using R_-, R_+.
   (a)

$$\left(\lim_{z \to -1-\infty\sqrt{-1}} \gamma^{(1)}(z), \lim_{z \to +1-\infty\sqrt{-1}} \gamma^{(2)}(z)\right) \in R_-. \tag{4.18}$$

   Here $\lim_{z \to -1-\infty\sqrt{-1}}$ is the limit when the imaginary part of z goes to $-\infty$ and $z \in \partial_1\Omega$. The meaning of $\lim_{z \to +1-\infty\sqrt{-1}}$ is similar.

(b)

$$(\lim_{z\to -1+\infty\sqrt{-1}} \gamma^{(1)}(z), \lim_{z\to +1+\infty\sqrt{-1}} \gamma^{(2)}(z)) \in R_+. \tag{4.19}$$

Here $\lim_{z\to -1+\infty\sqrt{-1}}$ is the limit when the imaginary part of z goes to $-\infty$ and $z \in \partial_1\Omega$. The meaning of $\lim_{z\to +1-\infty\sqrt{-1}}$ is similar.

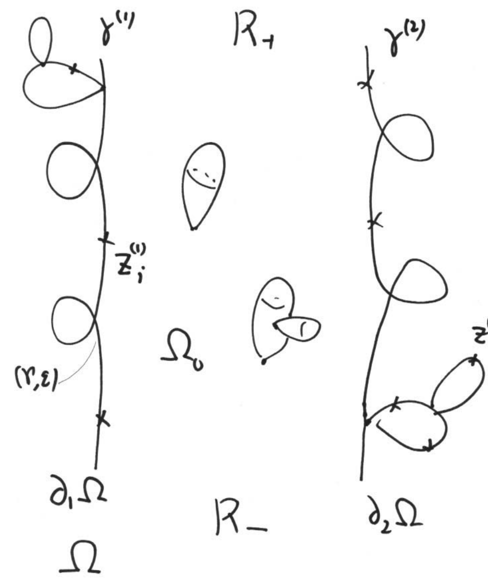

Figure 4.8

We say $(\Omega^1, u^1, \vec z^{1,(1)}, \vec z^{1,(2)})$ is equivalent to $(\Omega^2, u^2, \vec z^{2,(1)}, \vec z^{2,(2)})$ if there exists a homeomorphism $v : \Omega^1 \to \Omega^2$ such that

(1) v is biholomorphic on each irreducible component.
(2) $v(\partial_1\Omega^1) = \partial_1\Omega^2$.
(3) $v(z_j^{1,(1)}) = z_j^{2,(1)}$, $j = 1, \dots, k_1$ and $v(z_j^{1,(2)}) = z_j^{2,(2)}$, $j = 1, \dots, k_2$.
(4) $u^2 \circ v = u^1$.

The $\mathbb{R}$ action which translate the Imz direction of Ω is actually included in the definition of equivalence relation above.

We put

$$\begin{aligned}&\overset{\circ}{\mathcal{M}}_{k_1,k_2}(L_1, L_2; R_-, R_+; E)\\ &= \bigcup_{\vec i^{(1)};|\vec i^{(1)}|=k_1} \bigcup_{\vec i^{(2)};|\vec i^{(2)}|=k_2} \overset{\circ}{\mathcal{M}}(L_1, L_2; R_-, R_+; \vec i^{(1)}, \vec i^{(2)}; E)\end{aligned}$$

We define evaluation maps

$$\begin{aligned}&\mathrm{ev}^{(1)} : \overset{\circ}{\mathcal{M}}_{k_1,k_2}(L_1, L_2; R_-, R_+; E) \to (\tilde L_1 \times_Y \tilde L_1)^{k_1}\\ &\mathrm{ev}^{(2)} : \overset{\circ}{\mathcal{M}}_{k_1,k_2}(L_1, L_2; R_-, R_+; E) \to (\tilde L_2 \times_Y \tilde L_2)^{k_2}.\end{aligned} \tag{4.20}$$

They are defined by (4.16) and (4.17).

We also define the evaluation maps

$$\mathrm{ev}_i^{\infty} : \overset{\circ}{\mathcal{M}}_{k_1,k_2}(L_1, L_2; R_-, R_+; E) \to R_i \tag{4.21}$$

for $i = +, -$. They are defined by (4.18) and (4.19).

We consider the union of the fiber products:

$$\begin{aligned}
&\overset{\circ}{\mathcal{M}}_{k_1^0,k_2^0}(L_1, L_2; R_-, R_1; E_0) \times_{R_1} \\
&\overset{\circ}{\mathcal{M}}_{k_1^1,k_2^1}(L_1, L_2; R_1, R_2; E_1) \times_{R_2} \dots \\
&\dots \times_{R_{\ell-1}} \overset{\circ}{\mathcal{M}}_{k_1^{\ell-1},k_2^{\ell-1}}(L_1, L_2; R_{\ell-1}, R_\ell; E_{\ell-1}) \\
&\times_{R_\ell} \mathcal{M}_{k_1^\ell,k_2^\ell}(L_1, L_2; R_\ell, R_+; E_\ell),
\end{aligned} \tag{4.22}$$

where $k_0^{(j)} + k_1^{(j)} + \cdots + k_\ell^{(j)} = k_j$ for $j = 1, 2$, $E_0 + \cdots + E_\ell = E$, and R_i for $i = 1, \dots, \ell$ are connected components of $\tilde{L}_1 \times_Y \tilde{L}_2$. This union is by definition $\mathcal{M}_{k_1,k_2}(L_1, L_2; R_-, R_+; E)$. The evaluation maps (4.20) and (4.21) extends there.

Proposition 4.8. *We can define a topology on $\mathcal{M}_{k_1,k_2}(L_1, L_2; R_-, R_+; E)$ by which it becomes compact and Hausdorff.*

It has a Kuranishi structure with corner such that its boundary is a union of the following three types of fiber products.

(1)

$$\mathcal{M}_{k_1',k_2'}(L_1, L_2; R_-, R; E') \times_R \mathcal{M}_{k_1'',k_2''}(L_1, L_2; R, R_+; E'')$$

where $k_1' + k_1'' = k_1$, $k_2' + k_2'' = k_2$, $E' + E'' = E$ and R is a connected component of $\tilde{L}_1 \times_Y \tilde{L}_2$. See Figure 4.9.

(2)

$$\mathcal{M}_{k_1',k_2}(L_1, L_2; R_-, R_+; E')_{\mathrm{ev}_i^{(1)}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_1''}(L_1; E'')$$

where $k_1' + k'' + 1 = k_1$, $E' + E'' = E$ and $i = 1, \dots, k_1'$. The second factor is (2.29). (More precisely its analogue in immersed case.) See Figure 4.10.

(3)

$$\mathcal{M}_{k_1,k_2'}(L_1, L_2; R_-, R_+; E')_{\mathrm{ev}_i^{(2)}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_2''}(L_2; E'')$$

where $k_2' + k'' + 2 = k_2$, $E' + E'' = E$ and $i = 1, \dots, k_2'$. The second factor is (2.29). (More precisely its analogue in immersed case.) See Figure 4.11.

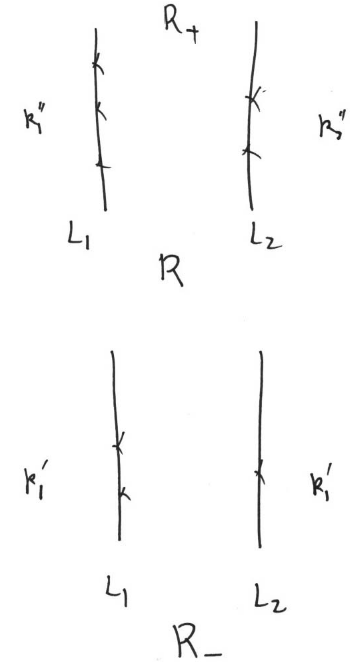

Figure 4.9

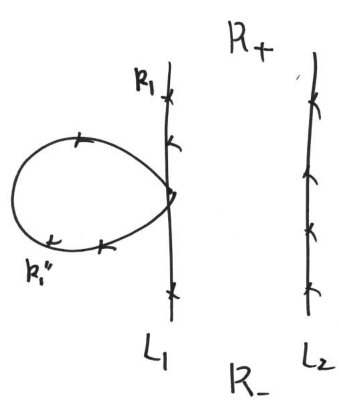

Figure 4.10

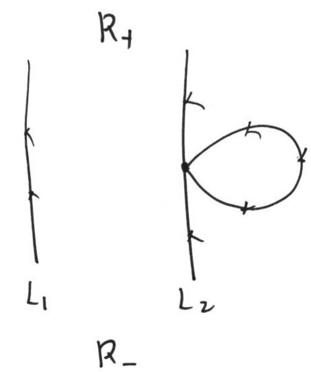

Figure 4.11

The proof of Proposition 4.8 is entirely similar to the proof of [FOOO2, Propositions 7.1.1, 7.1.2].

Let $x^1_1, \dots, x^1_{k_1} \in C(\tilde L_1 \times_Y \tilde L_1)$, $y_-, y_+ \in C(\tilde L_1 \times_Y \tilde L_2)$ and $x^2_1, \dots, x^2_{k_2} \in C(\tilde L_2 \times_Y \tilde L_2)$. We define:

$$\begin{aligned}
&\langle \mathfrak n_{k_1,k_2;E}(x^1_1,\dots,x^1_{k_1}; y_-; x^2_1,\dots,x^2_{k_2}), y_+\rangle \\
&= \#\big(\mathcal M_{k_1,k_2}(L_1,L_2;R_-,R_+;E) \;{}_{\mathrm{ev}^{(1)},\mathrm{ev}^\infty_-,\mathrm{ev}^{(2)},\mathrm{ev}^\infty_+}\times \\
&\qquad (x^1_1\times\dots\times x^1_{k_1}\times y_-\times x^2_1\times\dots\times x^2_{k_2}\times y_+)\big).
\end{aligned} \tag{4.23}$$

There are various chain models by which (4.23) becomes rigorous. (See [FOOO2], [FOOO5], [FOOO8], [FOOO9].) We will introduce a chain model which is also suitable to handle the moduli space appearing in gauge theory case in [Fu8], or in a separate paper. (We remark again we can work over $\mathbb Z_2$ because Y is assumed to be monotone. See [FOOO5].)

We then put

$$\begin{aligned}
\mathfrak n_{k_1,k_2} &= \sum T^E \mathfrak n_{k_1,k_2;E} \\
&: CF(L_1)^{k_1\otimes} \otimes_{\Lambda_0^{\mathbb Z_2}} CF(L_1,L_2) \otimes_{\Lambda_0^{\mathbb Z_2}} CF(L_2)^{k_2\otimes} \to CF(L_1,L_2).
\end{aligned} \tag{4.24}$$

Here

$$CF(L_i) = C(\tilde L_i \times_Y \tilde L_i; \Lambda_0^{\mathbb Z_2}), \qquad CF(L_1,L_2) = C(\tilde L_1 \times_Y \tilde L_2; \Lambda_0^{\mathbb Z_2}).$$

It is a part of the general theory of Kuranishi structure and virtual fundamental chain (See [FOOO9] for its thorough detail) that Proposition 4.8 and our definitions imply the following:

Corollary 4.9. *$\{\mathfrak n_{k_1,k_2}\}$ defines a structure of filtered A_∞ bimodule on $CF(L_1,L_2)$ over $(CF(L_1),\{\mathfrak m_k\})$-$(CF(L_2),\{\mathfrak m_k\})$.*

Namely we have

$$\begin{aligned}
0 &= \sum \mathfrak n_{k_1',k_2'}(x^1_1,\dots,\mathfrak n_{k_1'',k_2''}(x^1_{k_1'+1},\dots x^1_{k_1}; y; x^2_1,\dots,x^2_{k_2''});\dots,x^2_{k_2}) \\
&+ \sum \mathfrak n_{k_1',k_2}(x^1_1,\dots,\mathfrak m_{k_1''}(x_i,\dots,x_{i+k_1''-1}),\dots,x^1_{k_1}; y; x^2_1,\dots,x^2_{k_2}) \\
&+ \sum \mathfrak n_{k_1,k_2'}(x^1_1,\dots,x^1_{k_1}; y; x^2_1,\dots,\mathfrak m_{k_2''}(x_i,\dots,x_{i+k_2''-1}),\dots,x^2_{k_2}).
\end{aligned} \tag{4.25}$$

Here the sum in the first line is taken over k_1', k_1'', k_2', k_2'' with $k_1 = k_1' + k_1''$, $k_2 = k_2' + k_2''$. The sum in the second line is taken over k_1', k_1'', i with $k_1 + 1 = k_1' + k_1''$,

$i = 1, \dots, k_1' + 1$. *The sum in the third line is taken over* k_2', k_2'', i *with* $k_2 + 1 = k_2' + k_2''$, $i = 1, \dots, k_2' + 1$.

See [FOOO1, Formula (3.7.2)]. We remark that the first, second, third lines of Formula (4.25) corresponds to (1), (2), (3) in Proposition 4.8, respectively.

Let b_1, b_2 be bounding cochains of $CF(L_1)$, $CF(L_2)$, respectively. Following [FOOO1, Definition-Lemma 3.7.13] we define

$$\delta_{b_1,b_2} : CF(L_1, L_2) \to CF(L_1, L_2)$$

by

$$\delta_{b_1,b_2}(y) = \sum_{k_1,k_2} \mathfrak{n}_{k_1,k_2}(b_1, \dots, b_1; y; b_2, \dots, b_2). \tag{4.26}$$

(4.25) implies $\delta_{b_1,b_2} \circ \delta_{b_1,b_2} = 0$. ([FOOO1, Lemma 3.7.14].)

Definition 4.10. Floer homology of the pair $((L_1, b_1), (L_2, b_2))$ is

$$HF((L_1, b_1), (L_2, b_2)) = \frac{\operatorname{Ker} \delta_{b_1,b_2}}{\operatorname{Im} \delta_{b_1,b_2}}.$$

This is the Floer homology appearing in (1.1), (1.3), (4.5).

We finish digression and go back to the description of the compactifiation of the moduli space $\overset{\circ}{\mathcal{M}}_{k_1,k_3}((X, \mathcal{E}_X), L; R_1, R_2, R_3; E)$. By definition, ends of type (III) correspond to the following fiber product.

$$\overset{\circ}{\mathcal{M}}_{k_1',k_3'}((X, \mathcal{E}_X), L; R_1', R_2, R_3; E_1)_{\mathrm{ev}_1^\infty} \times_{\mathrm{ev}_-} \mathcal{M}_{k_1'',k_2''}(R(M), L; R_1', R_1; E_2). \tag{4.27}$$

Here $k_1' + k_1'' = k_1$, $E_1 + E_2 = E$ and R_1' is a connected component of $R(M) \times_{R(\Sigma)} \tilde{L}$.

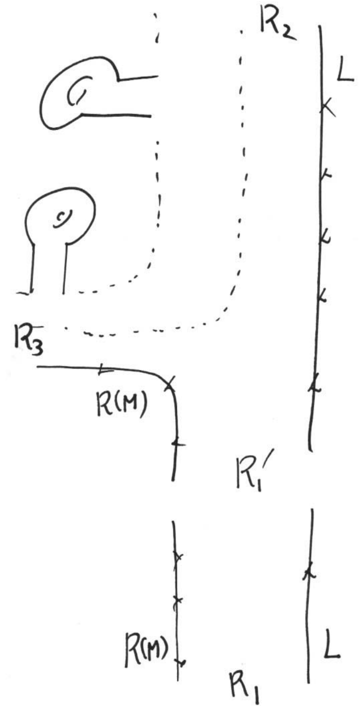

Figure 4.12

Definition 4.11. The set $\mathcal{M}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)$ is the union of the following fiber products:

$$\begin{aligned} &\overset{\circ}{\mathcal{M}}_{k_1^1,k_3^1}((X,\mathcal{E}_X),L;R_1',R_2',R_3';E_1)\\ &{}_{\mathrm{ev}_2^\infty}\times_{\mathrm{ev}_-}\mathcal{M}_{k_1^2}((M,\mathcal{E}),L;R_2',R_2;E_2)\\ &{}_{\mathrm{ev}_3^\infty}\times_{\mathrm{ev}_+}\mathcal{M}_{k_3^2}((M,\mathcal{E}),R(M);R_3,R_3';E_3)\\ &{}_{\mathrm{ev}_1^\infty}\times_{\mathrm{ev}_+}\mathcal{M}_{k_1^3,k_2^3}(R(M),L;R_1,R_1';E_4). \end{aligned} \tag{4.28}$$

Here $k_1^1+k_1^2+k_1^3=k_1$, $k_2^1+k_2^2+k_2^3=k_2$, $E_1+E_2+E_3+E_4=E$ and R_2', R_3', R_1' are connected components of $R(M)\times_{R(\Sigma)}R(M)$, $R(M)\times_{R(\Sigma)}R(M)$, $R(M)\times_{R(\Sigma)}\tilde{L}$, respectively.

We remark that we include the case when some of the 2nd, 3rd, 4th factors of the fiber product (4.28) is trivial. For example, if $E_2=0$, $R_2'=R_2$ and $k_1^2=0$ then the factor $\mathcal{M}_{k_1^2}((M,\mathcal{E}),L;R_2',R_2;E_2)$ drops.

Proposition 4.12. *We can define a topology on $\mathcal{M}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)$ so that it becomes compact and Hausdorff.*

The space $\mathcal{M}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)$ carries a virtual fundamental chain such that its boundary is the sum of the virtual fundamental chains of the following 5 types of fiber products.

(1) *The compactification of (4.13), which is:*

$$\mathcal{M}_{k_1',k_3}((X,\mathcal{E}_X),L;R_1,R_2',R_3;E_1)_{\mathrm{ev}_2^\infty}\times_{\mathrm{ev}_-}\mathcal{M}_{k_1''}((M,\mathcal{E}),L;R_2',R_2;E_2).$$

(2) *The compactification of (4.14), which is:*

$$\mathcal{M}_{k_1,k_3'}((X,\mathcal{E}_X),L;R_1,R_2,R_3';E_1)_{\mathrm{ev}_3^\infty} \times_{\mathrm{ev}_+} \mathcal{M}_{k_3''}((M,\mathcal{E}),R(M);R_3,R_3';E_2).$$

(3) *The compactification of (4.27), which is:*

$$\mathcal{M}_{k_1',k_3'}((X,\mathcal{E}_X),L;R_1',R_2,R_3;E_1)_{\mathrm{ev}_1^\infty} \times_{\mathrm{ev}_+} \mathcal{M}_{k_1'',k_2''}(R(M),L;R_1,R_1';E_2).$$

(4)

$$\mathcal{M}_{k_1',k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E_1)_{\mathrm{ev}_i^{(1)}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_1''}(L;E_2),$$

where $k_1'+k_1''=k_1+1$, $E_1+E_2=E$.

(5)

$$\mathcal{M}_{k_1,k_3'}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E_1)_{\mathrm{ev}_i^{(3)}} \times_{\mathrm{ev}_0} \mathcal{M}_{k_3''}(R(M);E_2),$$

where $k_3'+k_3''=k_3+1$, $E_1+E_2=E$.

Proof. (1)(2)(3) corresponds to the end of Type (I), (II), (III) respectively. (4) corresponds to the disk bubble on $\partial_1 W$. (See Figure 4.13.) (5) corresponds to the disk bubble on $\partial_3 W$. All other bubbles occur in codimension 2. □

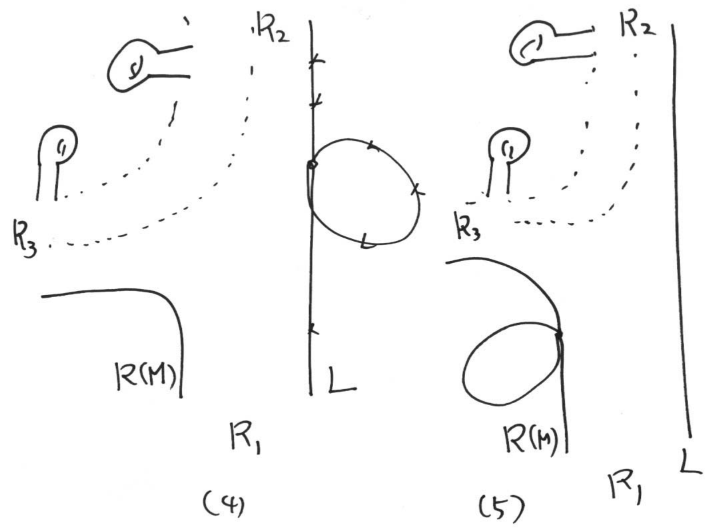

Figure 4.13

We remark again that $\mathcal{M}_{k_1,k_3}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)$ does not carry Kuranishi structure because Uhlenbeck compactification of the moduli space of ASD connections does not carry one. So we need to generalize the story of Kuranishi structure and its virtual fundamental chain. We will do it in [Fu8] or in a separate paper.

Now we consider $x_1^{(1)},\dots,x_{k_1}^{(1)} \in C(\tilde L \times_{R(\Sigma)} \tilde L;\Lambda_0^{\mathbb{Z}_1})$, $x_1^{(3)},\dots,x_{k_3}^{(3)} \in C(R(M)\times_{R(\Sigma)} R(M);\Lambda_0^{\mathbb{Z}_1})$, $y_1 \in C(R(M)\times_{R(\Sigma)} R(M);\Lambda_0^{\mathbb{Z}_1})$ $y_2,y_3 \in C(R(M)\times_{R(\Sigma)} \tilde L;\Lambda_0^{\mathbb{Z}_1})$. We define

$$\psi_{k_1,k_3} : CF(M,R(M))\otimes CF(R(M))^{k_3\otimes}\otimes CF(R(M),L)\otimes CF(L)^{k_1\otimes} \to CF(M,L)$$

by

$$\begin{aligned}&\langle\psi_{k_3,k_1}(y_2;x^{(3)}_1,\dots,x^{(3)}_{k_3};y_3;x^{(1)}_1,\dots,x^{(1)}_{k_1}),y_1\rangle\\&=\#\big(\mathcal{M}_{k_3,k_1}((X,\mathcal{E}_X),L;R_1,R_2,R_3;E)\\&\quad{}_{\mathrm{ev}^\infty_3,\mathrm{ev}^{(3)},\mathrm{ev}^\infty_1,\mathrm{ev}^{(1)},\mathrm{ev}^\infty_2}\times\\&\quad(y_2\times x^{(3)}_1\times\dots\times x^{(3)}_{k_3}\times y_3\times x^{(1)}_1\times\dots\times x^{(1)}_{k_1}\times y_1)\big).\end{aligned}\tag{4.29}$$

Lemma 4.13. *ψ_{k_1,k_3} satisfies the next equality.*

$$\begin{aligned}&\sum\mathfrak{n}_{k'_1}(\psi_{k_3,k''_1}(y_2;x^{(3)}_1,\dots,x^{(3)}_{k''_3};y_3;x^{(1)}_1,\dots,x^{(1)}_{k''_1}),x^{(1)}_{k''_1+1},\dots,x^{(1)}_{k_1})\\&+\sum\psi_{k''_3,k_1}(\mathfrak{n}_{k'_3}(y_2;x^{(3)}_1,\dots,x^{(3)}_{k'_3});x^{(3)}_{k'_3+1},\dots,x^{(3)}_{k_3};y_3;x^{(1)}_1,\dots,x^{(1)}_{k_1})\\&+\sum\psi_{k'_3,k'_1}(y_2;x^{(3)}_1,\dots,x^{(3)}_{k'_3},\mathfrak{n}_{k''_3,k''_1}(x^{(3)}_{k'_3+1},\dots,x^{(3)}_{k_3};y_3;x^{(1)}_1,\dots,x^{(1)}_{k''_1}),\dots,x^{(1)}_{k_1})\\&+\sum\psi_{k''_3,k_1}(y_2;x^{(3)}_1,\dots,\mathfrak{m}_{k'_3}(x^{(3)}_i,\dots,x^{(3)}_{i+k'_3-1}),\dots x^{(3)}_{k_3};y_3;x^{(1)}_1,\dots,x^{(1)}_{k_1})\\&+\sum\psi_{k_3,k''_1}(y_2;x^{(3)}_1,\dots,x^{(3)}_{k_3};y_3;x^{(1)}_1,\dots,\mathfrak{m}_{k'_1}(x^{(1)}_i,\dots,x^{(1)}_{i+k'_1-1}),\dots,x^{(1)}_{k_1})\\&=0.\end{aligned}$$

Here the sum in the first line is taken over k'_1,k''_1 with $k_1=k'_1+k''_1$ and $\mathfrak{n}_{k'_1}$ is the right filtered A_∞ module structure of $CF(M,L)$.

The sum in the second line is taken over k'_3,k''_3 with $k_3=k'_3+k''_3$ and $\mathfrak{n}_{k'_3}$ in the second line is the right filtered A_∞ module structure of $CF(M,R(M))$.

The sum in the third line is taken over k'_1,k''_1, k'_3,k''_3 with $k_1=k'_1+k''_1$, $k_3=k'_3+k''_3$ and $\mathfrak{n}_{k''_3,k''_1}$ in the third line is the filtered A_∞ bimodule structure on $CF(R(M),L)$.

The sum in the fourth line is taken over k'_3,k''_3,i with $k_3+1=k'_3+k''_3$, $i=1,\dots,k''_3$ and $\mathfrak{m}_{k'_3}$ in the fourth line is the filtered A_∞ algebra structure on $CF(R(M))$.

The sum in the fifth line is taken over k'_1,k''_1,i with $k_1+1=k'_1+k''_1$, $i=1,\dots,k''_1$ and $\mathfrak{m}_{k'_1}$ in the fourth line is the filtered A_∞ algebra structure on $CF(L)$.

Proof. This is a consequence of Proposition 4.12. In fact boundary components (1), (2), (3), (4), (5) in Proposition 4.12 corresponds to 1st, 2nd, 3rd, 4th, 5th lines of (4.29) respectively. □

We now define

$$\varphi_{k_1}:CF((R(M),b_M),L)\otimes CF(L)^{\otimes k_1}\to CF(M;L)\tag{4.30}$$

by

$$\varphi_{k_1}(y_3;x^{(1)}_1,\dots,x^{(1)}_{k_1})=\sum_{k_3}\psi_{k_3,k_1}(\mathbf{1}_M;\underbrace{b_M,\dots,b_M}_{k_3};y_3;x^{(1)}_1,\dots,x^{(1)}_{k_1}),\tag{4.31}$$

where $\mathbf{1}_M$ (resp. b_M) is as in Definition 3.7 (resp. Theorem 3.9).

We claim

$$\begin{aligned}&\sum\mathfrak{n}_{k''_1}(\varphi_{k'_1}(y_3;x^{(1)}_1,\dots,x^{(1)}_{k'_1}),\dots,k''_1)\\&+\sum\varphi_{k''_1}({}^{b_M}\mathfrak{n}_{k'_1}(y_3;x^{(1)}_1,\dots,x^{(1)}_{k'_1}),\dots,k''_1)\\&+\sum\mathfrak{n}_{k'_1}(y_3;x^{(1)}_1,\dots,\mathfrak{m}_{k''_1}(x_i,\dots,x_{i+k''_1-1}),\dots,x^{(1)}_{k_1})=0.\end{aligned}\tag{4.32}$$

In fact 1st, 2nd, 3rd term of (4.32) corresponds 1st, 2nd, 5th lines of the formula in Lemma 4.13, respectively. Note the sum of 3rd line of the formula in Lemma

4.13 vanish when we put $y_2 = \mathbf{1}_M$ and $x_i^{(3)} = b_M$, because of (3.9). Note the sum of 4th line of the formula in Lemma 4.13 vanish when we put $x_i^{(3)} = b_M$ since b_M is a bounding cochain.

(4.32) means that $\hat{\varphi} = \{\varphi_k\}$ consists a filtered A_∞ right module homomorphism

$$(CF((R(M), b_M), L), \{^{b_M}\mathfrak{n}_k\}) \to (CF(M; L), \{\mathfrak{n}_k\})).$$

To prove (4.4) we consider the moduli space $\mathcal{M}_{0,0}((X, \mathcal{E}_X), L; R_1, R_2, R_3; 0)$ in Definition 4.7. We recall that this space consists of $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u, \vec{z}^{(1)}, \vec{z}^{(3)})$ where $\mathfrak{A}$ is a flat connection, u is a constant map, $\Omega = (0, 1] \times \mathbb{R}$ and $\vec{z}^{(1)} = \vec{z}^{(3)} = \emptyset$. We also consider the case R_3 is the fundamental class, which is $\mathbf{1}_M$. Therefore the energy 0 part of φ_0 is the identity map.

The proof of Theorem 4.2 is complete. □

Remark 4.14. We remark that ψ_{k_1,k_3} induces a chain map

$$\psi : CF(M, (R(M), b_M)) \otimes CF((R(M), b_M), (L, b)) \to CF(M, (L, b)) \tag{4.33}$$

by using a bounding cochain b of L and b_M of $R(M)$. This map is very similar to the map $\mathfrak{m}_2$ defining the composition of morphisms in $\mathscr{FUK}(R(\Sigma))$. The only difference is: in (4.33) M plays a role of a 'Lagrangian submanifold' or an object of $\mathscr{FUK}(R(\Sigma))$.[6] If we regard M as an 'object' of $\mathscr{FUK}(R(\Sigma))$, then Theorem 1.3 claims that this object 'M' is isomorphic to the object $(R(M), b_M)$ in $\mathscr{FUK}(R(\Sigma))$. To prove this 'fact' we need to find an element of $CF(M, (R(M), b_M)$ which gives the isomorphism. The obvious candidate of such an element is the fundamental class $\mathbf{1}_M$. Our proof of Theorem 1.3 shows that we can choose b_M appropriately so that $\mathbf{1}_M$ indeed becomes an isomorphism.

Remark 4.15. The proof of Theorem 1.3 in this section has an analogue in the story of Wehrheim-Woodward functoriality. In fact we can use it to prove the next theorem.

Theorem 4.16. *Suppose we are in the situation Theorem 3.13. Then for any spin immersed Lagrangian submanifold L of M_2 and bounding cochain b of the filtered A_∞ algebra of L, we have a canonical isomorphism:*

$$HF((L_{12}, b_{12}), (L_1, b_1) \times (L, b)) \cong HF((L_2, b_2), (L, b)). \tag{4.34}$$

To prove Theorem 4.16 we replace Figure 4.9 by the following Figure 4.14.

[6] To regard M as an object of $\mathscr{FUK}(R(\Sigma))$ is an idea which was a starting point of the whole project of 'Floer homology of 3 manifolds with boundary' and was mentioned in [D4].

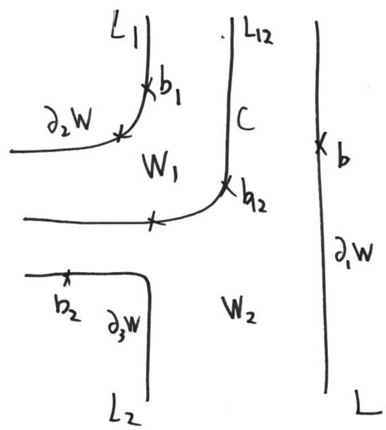

Figure 4.14

Here we consider a combination of pseudo-holomorphic maps u from W. Namely on W_1 we consider u as a map to M_1 and on W_2 we consider u as a map to M_2.

We put $C = W_1 \cap W_2$. We assume u maps C to L_{12}. We also require $u(\partial_1 W) \subset L$, $u(\partial_2 W) \subset L_1$ and $u(\partial_3 W) \subset L_2$. Here $\partial_2 W = \partial W_1 \setminus C$, $\partial_1 W \cup \partial_3 W = \partial W_2 \setminus C$.

We use bounding cochain b_1 on $\partial_2 W$, b_{12} on C, b_2 on $\partial_3 W$ and b on $\partial_1 W$. The role of $\mathbf{1}_M$ in the proof of Theorem 1.3 is taken by the fundamental class of $(L_1 \times L_2) \cap L_{12} \cong L_2$.

We remark that Figure 4.14 is similar to [LL, Figure 1]. In fact in case L_1, L_{12}, L_2 and L are all monotone, Figure 4.14 is a special case of [LL, Figure 1]. In fact [LL] discussed the case of composition of Lagrangian correspondences L_{01} from M_0 to M_1 and L_{12} from M_1 to M_2. If we put M_0 =point, then it corresponds to our situation. I also like to mention that the result of [LL] obtained by [LL, Figure 1] is a variant of an earlier results in [WW1] and the map $u : (W_1, W_2) \to (M_1, M_1 \times M_2)$ is a particular case of objects called pseudo-holomorphic quilt in [WW1].

5. Gluing isomorphism of relative Floer homology

In this section we prove Theorem 1.1 (2). The proof is based on a similar idea as the proof of Theorem 1.3 in Section 4. We consider $(M_1, \mathcal{E}_1)$ and $(M_2, \mathcal{E}_2)$ as in Theorem 1.1 (2). Let $(M, \mathcal{E})$ be a closed 3-manifold with $SO(3)$ bundle obtained by gluing $(M_1, \mathcal{E}_1)$ and $(M_2, \mathcal{E}_2)$. We fix a Riemannian metric on M. We will construct a chain map

$$\Phi : CF((R(M_1), b_{M_1}), (R(M_2), b_{M_2})) \to CF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2}) \tag{5.1}$$

which is congruent to the identity map modulo $\Lambda_+^{\mathbb{Z}_2}$. Here the chain complex $CF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2})$ is the Floer's chain complex which defines $SO(3)$-Floer homology of $(M, \mathcal{E})$. We first review its definition from [Fl1], [Fl2].

We assume that $R(M_1)$ is transversal to $R(M_2)$ for simplicity. (We can remove this assumption by appropriate perturbation.) Then the set of flat connections on $(M, \mathcal{E})$ is a finite set and is identified with $R(M_1) \cap R(M_2)$. We denote this set by $R(M)$.

Definition 5.1. Let $a_-, a_+ \in R(M)$. We denote by $\widetilde{\overset{\circ\circ}{\mathcal{M}}}(a_-, a_+)$ the gauge equivalence classes of all connections $\mathfrak{A}$ on $(M \times \mathbb{R}, \mathcal{E} \times \mathbb{R})$ such that the following holds.

(1) $\mathfrak{A}$ is an ASD connection. Namely it satisfies the equation (2.12).
(2) The energy $\mathcal{E}(\mathfrak{A})$ is finite. Here

$$\mathcal{E}(\mathfrak{A}) = \int_{M\times\mathbb{R}} \|F_{\mathfrak{A}}\|^2 \Omega_{\mathbf{g}}.$$

(3) We put the following asymptotic boundary condition. There exists a gauge transformation g such that:

$$\lim_{t\to\infty} g^*\mathfrak{A}|_{M\times\{t\}} = a_+, \qquad \lim_{t\to-\infty} g^*\mathfrak{A}|_{M\times\{t\}} = a_-. \tag{5.2}$$

Remark 5.2. Floer [Fl1] proved the following. The transversality of $R(M_1)$ and $R(M_2)$ implies that the convergence in (5.2) is of exponential order in the sense of (2.17). (See also Remark 2.5.)

Floer also proved the following if $\mathfrak{A}$ be a connection satisfying (1)(2) above. Then there exists a_-, a_+ such that (3) is satisfied.

We divide $\widetilde{\overset{\circ\circ}{\mathcal{M}}}(a_-, a_+)$ by the $\mathbb{R}$ action defined by translation and denote by $\overset{\circ\circ}{\mathcal{M}}(a_-, a_+)$ the quotient space. Including bubble in the same way as Uhlenbeck compactification we obtain $\overset{\circ}{\mathcal{M}}(a_-, a_+)$. Finally we define $\mathcal{M}(a_-, a_+)$ as the union of

$$\overset{\circ}{\mathcal{M}}(a_-, a_1) \times \overset{\circ}{\mathcal{M}}(a_1, a_2) \times \cdots \times \overset{\circ}{\mathcal{M}}(a_k, a_+), \tag{5.3}$$

where $a_1, \dots, a_k \in R(M)$. Then we can define a topological space $\mathcal{M}(a_-, a_+)$ so that it becomes compact and Hausdorff.

We now define

Definition 5.3. Let $CF(M, \mathcal{E})$ be a $\mathbb{Z}_2$ vector space where the set of its basis is identified with $R(M)$.

The Floer boundary operator $\partial : CF(M, \mathcal{E}) \to CF(M, \mathcal{E})$ is defined by:

$$\partial[a_-] = \sum_{a_+} \#(\mathcal{M}(a_-, a_+))[a_+], \tag{5.4}$$

where the sum is taken over all $a_+ \in R(M)$ and a component of $\mathcal{M}(a_-, a_+)$ such that the virtual dimension of $\mathcal{M}(a_-, a_+)$ is zero.

By using the moduli space $\mathcal{M}(a_-, a_+)$ in case its virtual dimension is 1 we can prove $\partial \circ \partial = 0$. We then put:

$$HF(M, \mathcal{E}) = \frac{\mathrm{Ker}\partial}{\mathrm{Im}\partial} \tag{5.5}$$

and call it the $SO(3)$-*Floer homology.*

Since in Lagrangian Floer theory, it is standard to use the universal Novikov ring $\Lambda_0^{\mathbb{Z}_2}$ as its coefficient ring, for the sake of consistency, we define a variant $HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2})$ of $HF(M, \mathcal{E})$ which is a module over $\Lambda_0^{\mathbb{Z}_2}$.

We denote by $\mathcal{M}(a_-, a_+)_0$ the component of $\mathcal{M}(a_-, a_+)$ of virtual dimension 0. Using the monotonicity, we find that the energy $\mathcal{E}(\mathfrak{A})$ of elements $\mathfrak{A}$ of $\mathcal{M}(a_-, a_+)_0$

is independent of $\mathfrak{A}$. We write it $\mathcal{E}(a_-, a_+; 0)$. (This number is not defined if $\mathcal{M}(a_-, a_+)_0$ is an empty set.) We put

$$CF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2}) = CF(M, \mathcal{E}) \otimes_{\mathbb{Z}_2} \Lambda_0^{\mathbb{Z}_2} \tag{5.6}$$

and define $\partial' : CF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2}) \to CF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2})$ by:

$$\partial'[a_-] = \sum_{a_+} T^{\mathcal{E}(a_-, a_+; 0)} \#(\mathcal{M}(a_-, a_+))[a_+]. \tag{5.7}$$

The proof by Floer of the equality $\partial \circ \partial = 0$ can be used without change to show $\partial' \circ \partial' = 0$. We now define:

Definition 5.4.

$$HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2}) = \frac{\mathrm{Ker}\partial'}{\mathrm{Im}\partial'}. \tag{5.8}$$

Remark 5.5. This remark is not related to the proof of Theorem 1.1 (2) so much but is related to the point that $\Lambda_0^{\mathbb{Z}_2}$ coefficient version might have some more information than $\mathbb{Z}_2$ coefficient version.

(1) Note $HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2}) \ne HF(M, \mathcal{E}) \otimes_{\mathbb{Z}_2} \Lambda_0^{\mathbb{Z}_2}$. In fact it is easy to see that

$$\mathrm{rank}_{\mathbb{Z}_2} \frac{HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2})}{\Lambda_+^{\mathbb{Z}_2} HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2})} = \#R(M),$$

which can be in general different from $\mathrm{rank}_{\mathbb{Z}_2} HF(M, \mathcal{E})$. This is so in case Floer's boundary operator is nontrivial.

We use $\mathbb{Z}_2$ coefficient here since we use it in the main theorems of this paper. However we can certainly definite $HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}})$ since orientation and sigh is fully worked out in gauge theory Floer homology by Floer.

(2) In case $R(M)$ is not a finite set (or is not Fredholm regular), we need to perturb. The independence of $HF(M, \mathcal{E}; \Lambda_0^{\mathbb{Z}_2})$ of the perturbation does *not* hold. This is similar to the following fact: The Floer homology of a pair of Lagrangian submanifolds $HF((L_1, b_1), (L_2, b_2); \Lambda_0)$ over Λ_0 coefficient is *not* an invariant of Hamiltonian perturbation of L_1, L_2. (Namely for a pair of Hamiltonian diffeomorphisms φ_1, φ_2 of our symplectic manifold, the isomorphism

$$HF((\varphi_1(L_1), (\varphi_1)_*(b_1)), (\varphi_2(L_2), (\varphi_2)_*(b_2)); \Lambda_0) \cong HF((L_1, b_1), (L_2, b_2); \Lambda_0)$$

is false.) On the other hand, if we use Λ instead of Λ_0 as a coefficient ring then $HF((L_1, b_1), (L_2, b_2); \Lambda)$ becomes invariant of Hamiltonian isotopy. (See [FOOO1, Theorem 4.1.4] and [FOOO1, Theorem 4.1.5].) So there is an issue for the well-defined-ness of $HF(M, \mathcal{E}; \Lambda_0)$ in case $R(M)$ is not Fredholm regular. Nevertheless we can use the argument of [FOOO1, Section 6.5.4] to define $HF(M, \mathcal{E}; \Lambda_0^F)$ also in case $R(M)$ is not Fredholm regular, if F is a field.

The Floer homology $HF(M, \mathcal{E}; \Lambda_0^F)$ in general contain a torsion subgroup such as $\Lambda_0^F / T^{\lambda} \Lambda_0^F$ as its direct factor. It is not clear for the author whether such components can be applicable to the study of topology of 3 or 4 manifolds.

Let us go back to the proof of Theorem 1.1 (2). The chain complex in the right hand side of (5.1) is one defined by (5.6) and (5.7). The main part of the construction of (5.1) is the definition of the moduli space we use for that purpose.

We take a domain $W \subset \mathbb{C}$ such that the following holds. (See Figure 5.1.)

Condition 5.6. (1) The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im}z < -2\}$ is $\{z \in \mathbb{C} \mid |\mathrm{Re}z| \le 1, \mathrm{Im}z < -2\}$. The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im}z > +2\}$ is $\{z \in \mathbb{C} \mid |\mathrm{Re}z| \le 1, \mathrm{Im}z > +2\}$.

(2) The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im}z > +2\}$ is $\{z \in \mathbb{C} \mid |\mathrm{Re}z| \le 1, \mathrm{Im}z > +2\}$. The intersection $W \cap \{z \in \mathbb{C} \mid \mathrm{Im}z < -2\}$ is $\{z \in \mathbb{C} \mid |\mathrm{Re}z| \le 1, \mathrm{Im}z < -2\}$.

(3) The boundary ∂W has four connected components $\partial_i W$ $(i = 1, 2, 3, 4)$ each of which is a C^∞ submanifold of $\mathbb{C}$ and is diffeomorphic to $\mathbb{R}$. Moreover $\partial_1 W = \{z \in \mathbb{C} \mid \mathrm{Re}z > 0, \mathrm{Im}z > 0\}$, $\partial_2 W \subset \{z \in \mathbb{C} \mid \mathrm{Re}z < 0, \mathrm{Im}z > 0\}$, $\partial_3 W \subset \{z \in \mathbb{C} \mid \mathrm{Re}z < 0, \mathrm{Im}z < 0\}$. $\partial_4 W \subset \{z \in \mathbb{C} \mid \mathrm{Re}z > 0, \mathrm{Im}z < 0\}$.

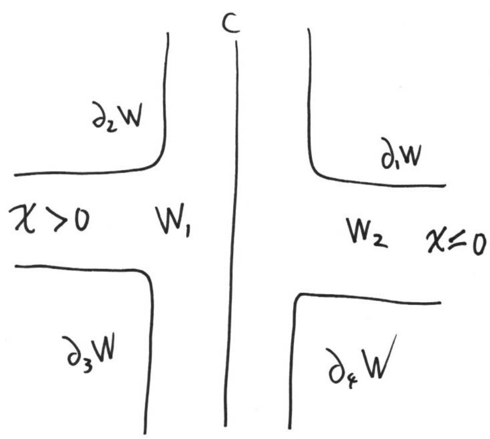

Figure 5.1

We put $C = \{z \mid \mathrm{Re}z = 0\} \subset W$ and

$$W_1 = \{z \in W \mid \mathrm{Re}z < 0\}, \qquad W_2 = \{z \in W \mid \mathrm{Re}z > 0\}.$$

We take $\chi : W \to [0,1]$ and a Riemannian metric $\mathbf{g}$ on $\Sigma \times W_1$ with the following properties. (See Figure 5.2.)

Condition 5.7. (1) $\{z \in \mathbb{C} \mid \chi(z) > 0\} = W_1$.

(2) On $\{z \in W \mid |\mathrm{Im}z| > -3\}$, $\chi(z) = \chi(\mathrm{Re}z)$, where χ in the right hand side is the same function as one appeared in (2.11). $\mathbf{g} = \chi^2 g_\Sigma + ds^2 + dt^2$ on $\{z \in W \mid |\mathrm{Im}z| > -3\}$, where we put $z = s + \sqrt{-1}t$.

(3) On $\{z \in W \mid \mathrm{Re}z < -3\}$, $\chi(z) = 1$. $\mathbf{g} = g_\Sigma + ds^2 + dt^2$ on $\Sigma \times \{z \in W \mid \mathrm{Re}z < -3\}$, where we put $z = s + \sqrt{-1}t$.

(4) In a neighborhood of $\Sigma \times \partial_2 W$ (resp. $\Sigma \times \partial_3 W$), the space $\Sigma \times W_1$ with metric $\mathbf{g}$ is isometric to the direct product $g_\Sigma \times (0, \epsilon) \times \mathbb{R}$. Here $g_\Sigma \times \{0\} \times \mathbb{R}$ corresponds to $\Sigma \times \partial_2 W$ (resp. $\Sigma \times \partial_3 W$). This isometry is compatible with the isometry obtained by items (2)(3) in the domain described by those items.

(5) On a $(-\epsilon, 0) \times \mathbb{R}$, $\chi(z) = \chi(\mathrm{Re}z)$, where χ in the right hand side is the same function as one appeared in (2.11). On $\Sigma \times (-\epsilon, 0) \times \mathbb{R}$ we have $\mathbf{g} = \chi^2 g_\Sigma + ds^2 + dt^2$ where we put $z = s + \sqrt{-1}t$.

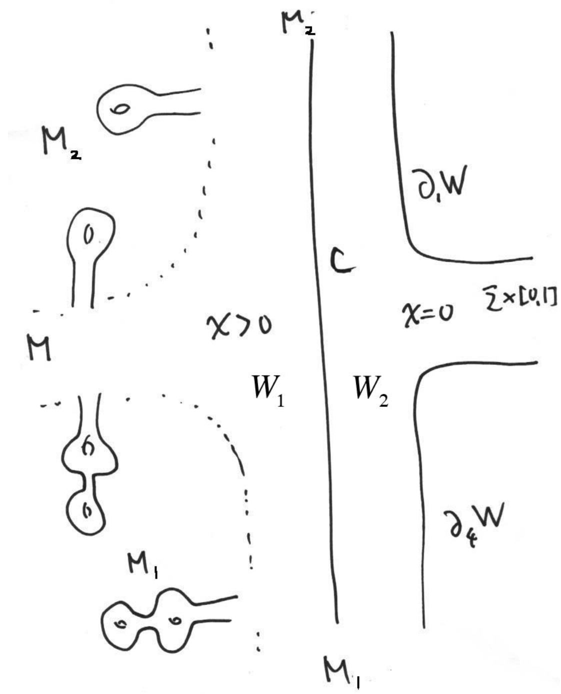

Figure 5.2

We extend the metric $\mathbf{g}$ on $\Sigma \times W_1$ to a 'singular metric' on $\Sigma \times W$ by putting $\mathbf{g} = 0 g_\Sigma + ds^2 + dt^2$ outside $\Sigma \times W_1$.

By Condition 5.7 $(\Sigma \times W_1, \mathbf{g})$ is isometric to $(\Sigma \times (0,\epsilon) \times \mathbb{R}, g_\Sigma + ds^2 + dt^2)$ at a neighborhood of $\partial_2 W$ (resp. $\partial_3 W$). We remark that $M_{2,0} \times \mathbb{R}$ (resp. $M_{1,0} \times \mathbb{R}$) is isometric to $(\Sigma \times (-\epsilon, 0) \times \mathbb{R}, g_\Sigma + ds^2 + dt^2)$ at a neighborhood of its boundary. Here $M_{1,0} = M_1 \setminus (\Sigma \times (-1,1])$, (resp. $-M_{2,0} = -M_2 \setminus (\Sigma \times (-1,1])$). Therefore we can glue them together to obtain $(Y, \mathbf{g})$. Here $\mathbf{g}$ is a 'Riemannian metric' which is degenerate on $\Sigma \times \overline{W_2}$.

We remark that Y has 4 ends and 2 boundary components. The 4 ends corresponds to $\mathrm{Re} z \to \pm\infty$ and $\mathrm{Im} z \to \pm\infty$. The end corresponding to $\mathrm{Re} z \to +\infty$ is of the form $\Sigma \times [-1,1] \times [c,\infty)$. The end corresponding to $\mathrm{Im} z \to +\infty$ is of the form $-M_2 \times [-1,1] \times [c,\infty)$. The end $\mathrm{Re} z \to -\infty$ is of the form $M \times (-\infty, -c]$. The end corresponding to $\mathrm{Im} z \to -\infty$ is of the form $M_1 \times (-\infty, -c]$.

The boundaries are $\Sigma \times \partial_1 W$ and $\Sigma \times \partial_4 W$. We remark that we glued $M_{2,0} \times \mathbb{R}$ (resp. $M_{1,0} \times \mathbb{R}$) to $\Sigma \times \partial_2 W$ (resp. $\Sigma \times \partial_3 W$). So $\Sigma \times \partial_2 W$ and $\Sigma \times \partial_3 W$ are not boundary components of Y.

The $SO(3)$ bundles $\mathcal{E}_1, \mathcal{E}_2$ on $M_1, -M_2$ induce an $SO(3)$ bundle on Y in an obvious way, which we denote by $\mathcal{E}_Y$. For a smooth connection $\mathfrak{A}$ of $\mathcal{E}_Y$ we can consider the 'ASD-equation'. Namely we require (2.12) on $Y \setminus (\Sigma \times \overline{W_2})$ and (2.14) on $\Sigma \times W \subset Y$. (Note (2.12) coincides with (2.14) on the overlapped part.) We say $\mathfrak{A}$ is an ASD-connection by an abuse of notation if it satisfies (2.12) on $Y \setminus (\Sigma \times \overline{W_2})$ and (2.14) on $\Sigma \times W \subset Y$.

We consider also Ω which contains W_2. Namely Ω is a union of W_2 and trees of disk and sphere components attached to ∂W_2 and $\mathrm{Int} W_2$, respectively. We consider

the pair (Ω, u) which satisfies Condition 2.7, except we replace $(0,1)\times\mathbb{R}$ by W_2. We call this condition Condition 2.7".

Now we modify Definitions 2.9, 2.23 and 5.8 as follows. We consider the decompositions (2.37) and (2.36). Let $I(R(M_j))$ be the index sets as in there. Namely

$$R(M_j)\times_{R(\Sigma)} R(M_j) = \bigcup_{k\in I(R(M_j))} R(M_j)_k \qquad j=1,2$$
$$R(M_1)\times_{R(\Sigma)} R(M_2) = R(M).$$

Let R_k $j=1,2$ be connected components of $R(M_j)\times_{R(\Sigma)} R(M_j)$ and $a_-, a_+ \in R(M) = R(M_1)\cap R(M_2)$.

We also take $\vec{i}^{(1)}$, $\vec{i}^{(2)}$ as in Definition 5.8 (5) below.

Definition 5.8. We define the set $\overset{\circ}{\mathcal{M}}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;\vec{i}^{(1)},\vec{i}^{(2)};E)$ as the set of all equivalence classes of $(\mathfrak{A},\mathfrak{z},\mathfrak{w},\Omega,u,\vec{z}^{(1)},\vec{z}^{(2)})$ satisfying the following conditions. (See Figure 5.3.)

(1) $\mathfrak{A}$ is a connection of $\mathcal{E}_Y$ satisfying equations (2.12), (2.14).
(2) $\mathfrak{z} = (\mathfrak{z}_1,\dots,\mathfrak{z}_{m_1})$ is an *unordered* m_1-tuple of points of $Y\setminus(\Sigma\times\overline{W_2})$. We put $\|\mathfrak{z}\| = m_1$. We say the subset $\{\mathfrak{z}_1,\dots,\mathfrak{z}_{m_1}\}\subset Y\setminus(\Sigma\times\overline{W_2})$ the *support* of $\mathfrak{z}$ and denote it by $|\mathfrak{z}|$. We define multi : $|\mathfrak{z}|\to\mathbb{Z}_{>0}$ by $\mathrm{multi}(x) = \#\{i \mid z_i = x\}$ and call it the *multiplicity function.*
(3) $\mathfrak{w} = (\mathfrak{w}_1,\dots,\mathfrak{w}_{m_2})$ is an *unordered* m_2-tuple of points of C. We put $\|\mathfrak{w}\| = m_2$. We say the subset $\{\mathfrak{w}_1,\dots,\mathfrak{w}_{m_2}\}\subset C$ the *support* of $\mathfrak{w}$. We define multi : $|\mathfrak{w}|\to\mathbb{Z}_{>0}$ by $\mathrm{multi}(x) = \#\{i \mid w_i = x\}$ and call it the *multiplicity function.*
(4) Ω satisfies Condition 2.7".
(5) $\vec{i}^{(1)} = (i^{(1)}(1),\dots,i^{(1)}(k_1)) \in I(R(M_1))^{k_1}$ and $\vec{i}^{(2)} = (i^{(2)}(1),\dots,i^{(2)}(k_2)) \in I(R(M_2))^{k_2}$
(6) $\vec{z}^{(1)} = (z^{(1)}_1,\dots,z^{(1)}_{k_1})$ (resp. $\vec{z}^{(2)} = (z^{(2)}_1,\dots,z^{(2)}_{k_2})$). $z^{(1)}_i$ lies on $\partial_4\Omega$, (resp. $z^{(2)}_i$ lies on $\partial_1\Omega$). None of $z^{(1)}_i$ or $z^{(2)}_i$ is a nodal point. If $i\neq j$ then $z^{(1)}_i \neq z^{(1)}_j$, $z^{(2)}_i\neq z^{(2)}_j$. $(z^{(1)}_1,\dots,z^{(1)}_{k_1})$ (resp. $(z^{(2)}_1,\dots,z^{(2)}_{k_2})$) respects counter clockwise orientation of $\partial_4\Omega$ (resp. $\partial_1\Omega$).
(7) There exists a smooth map $\gamma^{(1)}: \partial_4\Omega\setminus\{z^{(1)}_1,\dots,z^{(1)}_{k_1}\}\to R(M_1)$ such that $u(z) = i_{R(M_1)}(\gamma(z))$ on $\partial_4 W\setminus\{z^{(1)}_1,\dots,z^{(1)}_{k_1}\}$.

There exists a smooth map $\gamma^{(2)}: \partial_1\Omega\setminus\{z^{(2)}_1,\dots,z^{(2)}_{k_2}\}\to R(M_2)$ such that $u(z) = i_{R(M_2)}(\gamma(z))$ on $\partial_4 W\setminus\{z^{(2)}_1,\dots,z^{(2)}_{k_2}\}$.
(8) For $j=1,\dots,k_1$ the following holds.

$$(\lim_{z\uparrow z^{(1)}_j}\gamma^{(1)}(z), \lim_{z\downarrow z^{(1)}_j}\gamma^{(1)}(z)) \in \widehat{R(M_1)}_{i^{(1)}(j)}. \tag{5.9}$$

Here the notation $z\uparrow z_j$, $z\downarrow z_j$ is defined in the same way as (2.38)

For $j=1,\dots,k_2$ the following holds.

$$(\lim_{z\uparrow z^{(2)}_j}\gamma^{(2)}(z), \lim_{z\downarrow z^{(2)}_j}\gamma^{(2)}(z)) \in \widehat{R(M_2)}_{i^{(2)}(j)}. \tag{5.10}$$

Here the notation $z\uparrow z_j$, $z\downarrow z_j$ is defined in the same way as (2.38)

(9) We replace Condition 2.8 (3) by the stability of $(\Omega, u, \vec{z}^{(1)}, \vec{z}^{(2)})$. Namely the set of all maps $v : \Omega \to \Omega$ satisfying the next three conditions is a finite set.
   (a) v is a homeomorphism and is holomorphic on each of the irreducible components.
   (b) v is the identity map on $(0,1] \times \mathbb{R} \subseteq \Omega$.
   (c) $u \circ v = u$.
   (d) $v(z_j^{(1)}) = z_j^{(1)}$, $j = 1, \dots, k_1$ and $v(z_j^{(2)}) = z_j^{(2)}$, $j = 1, \dots, k_3$.

(10) For $(s,t) \in W_2$ we have
$$[A(s,t)] = u(s,t).$$
Here $A(s,t)$ is obtained from $\mathfrak{A}$ by (2.13).

(11) The energy of $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ which is defined in the same way as Definition 2.10 is E.

(12) We assume the following asymptotic boundary conditions, which are defined by using R_1, R_2, a_-, a_+.
   (a)
$$\lim_{z \to +\infty - \sqrt{-1}} \gamma^{(4)}(z) = \lim_{z \to +\infty + \sqrt{-1}} \gamma^{(1)}(z) = a_+. \tag{5.11}$$
   Here $\lim_{z \to +\infty - \sqrt{-1}}$ is the limit when the real part of $z \in \partial_4 W$ goes to $+\infty$. The meaning of $\lim_{z \to +\infty + \sqrt{-1}}$ is similar.
   (b) We consider the restriction of $\mathfrak{A}$ to $\Sigma \times \{z \in W \mid \mathrm{Re} z = -c\}$ for $c > 3$. We glue it with the restriction $\mathfrak{A}$ to $(M_1)_0 = M_1 \setminus (\Sigma \times (-1,1])$ and $(M_2)_0 = M_2 \setminus (\Sigma \times (-1,1])$ which are attached to $\Sigma \times \{-c - \sqrt{-1}\}$ and $\Sigma \times \{-c + \sqrt{-1}\}$ respectively. (See Figure 5.4.) We call it $\mathfrak{A}|_{\mathrm{Re} z = -c}$. It is a connection on $M = M_1 \cup -M_2$. We assume that $\mathfrak{A}|_{\mathrm{Re} z = -c}$ converges to a flat connection a_- as $c \to \infty$.
   (c) We consider the restriction of $\mathfrak{A}$ to $\Sigma \times \{z \in W \mid \mathrm{Im} z = c\}$ for $c > 3$. We glue it with the restriction $\mathfrak{A}$ to $-M_2 = -M_2 \setminus (\Sigma \times (-1,1])$, which is attached to $-1 + c\sqrt{-1}$. (See Figure 5.5.) We call it $\mathfrak{A}|_{\mathrm{Im} z = c}$. It is a connection of $-M_2$. We assume that $\mathfrak{A}|_{\mathrm{Im} z = c}$ converges to a flat connection as $c \to +\infty$. We write its limit $\lim_{c \to +\infty} \mathfrak{A}|_{\mathrm{Im} z = c}$.
   Then we also assume
$$(\lim_{c \to +\infty} \mathfrak{A}|_{\mathrm{Im} z = c}, \lim_{z \to +1 + \infty\sqrt{-1}} \gamma^{(1)}(z) \in R_2. \tag{5.12}$$
   Here the meaning of $\lim_{z \to +1 + \infty\sqrt{-1}}$ is similar to (5.11).
   (d) We consider the restriction of $\mathfrak{A}$ to $\Sigma \times \{z \mid \mathrm{Im} z = -c\}$ for $c > 3$. We glue it with the restriction $\mathfrak{A}$ to $(M_1)_0$ which is attached to $-c + \sqrt{-1}$. (See Figure 5.6.) We call it $\mathfrak{A}|_{\mathrm{Im} z = -c}$. It is a connection of M_1. We assume that $\mathfrak{A}|_{\mathrm{Im} z = -c}$ converges to a flat connection as $c \to \infty$. We write its limit $\lim_{c \to \infty} \mathfrak{A}|_{\mathrm{Im} z = -c}$.
   Then we also assume
$$(\lim_{c \to \infty} \mathfrak{A}|_{\mathrm{Im} z = -c}, \lim_{z \to -\infty - \sqrt{-1}} \gamma^{(3)}(z)) \in R_1. \tag{5.13}$$
   Here the meaning of $\lim_{z \to -\infty - \sqrt{-1}}$ is similar to (5.11).

The equivalence relation is defined in the same way as Definition 2.11.

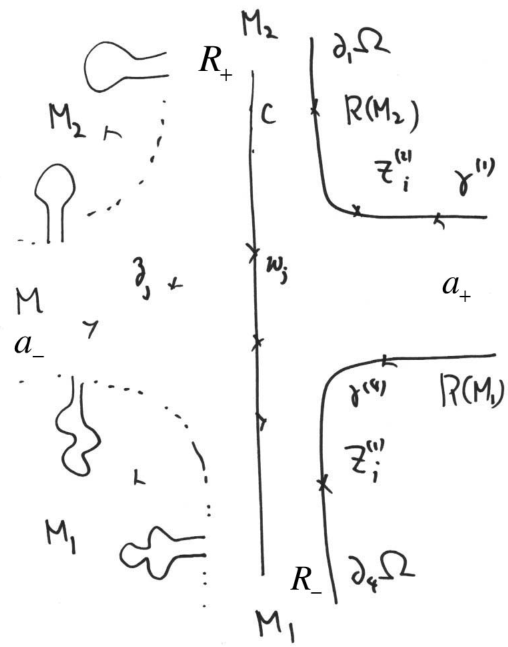

Figure 5.3

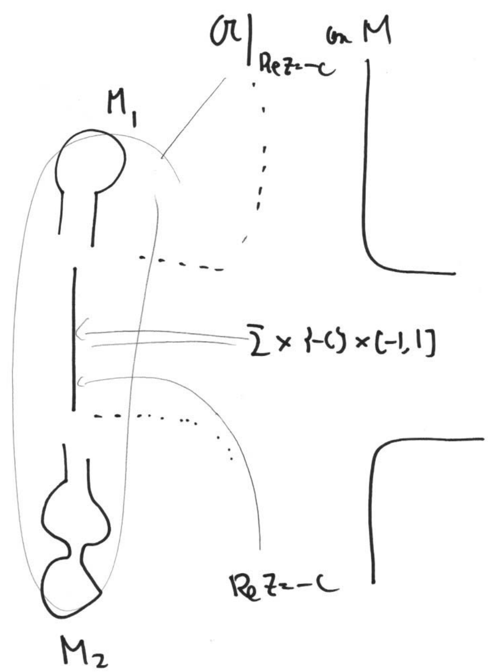

Figure 5.4

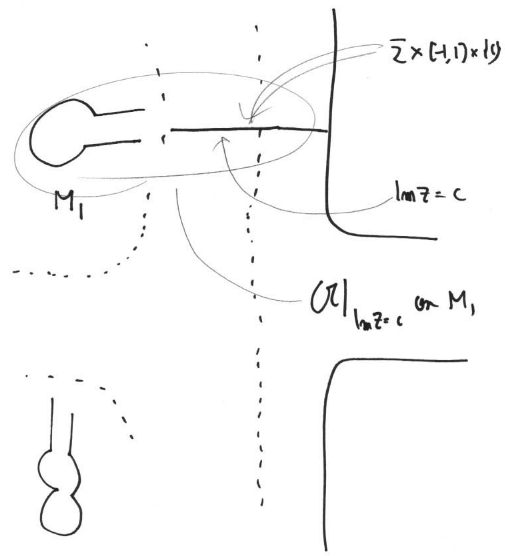

Figure 5.5

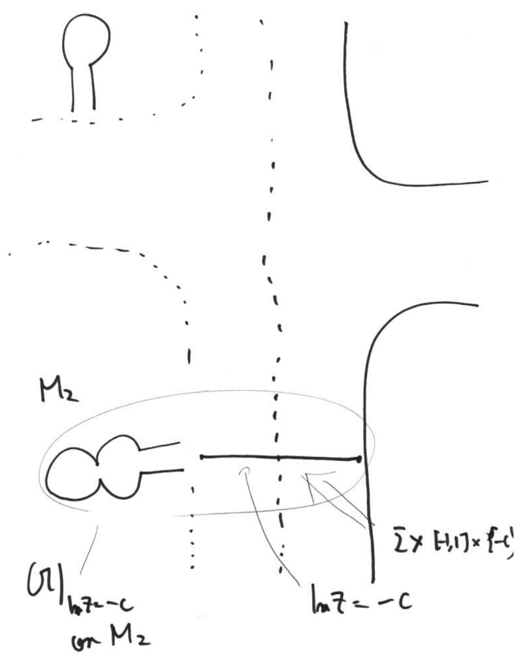

Figure 5.6

We can define a topology on $\overset{\circ}{\mathcal{M}}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;\vec{i}^{(1)},\vec{i}^{(2)};E)$ by modifying Definition 2.12 in an obvious way.

We put

$$
\begin{aligned}
&\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)\\
&=\bigcup_{\vec{i}^{(1)};|\vec{i}^{(1)}|=k_1}\ \bigcup_{\vec{i}^{(2)};|\vec{i}^{(2)}|=k_2}\overset{\circ}{\mathcal{M}}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;\vec{i}^{(1)},\vec{i}^{(2)};E)
\end{aligned}
$$

$\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)$ is a Hausdorff space.

We define evaluation maps

$$
\begin{aligned}
&\mathrm{ev}^{(1)}:\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)\to(R(M_1)\times_{R(\Sigma)}R(M_1))^{k_1}\\
&\mathrm{ev}^{(2)}:\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)\to(R(M_2)\times_{R(\Sigma)}R(M_2))^{k_2}.
\end{aligned}
\tag{5.14}
$$

They are defined by (5.9) and (5.10), respectively.

We also define the evaluation maps

$$
\mathrm{ev}_i^{\infty}:\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)\to R_i \tag{5.15}
$$

for $i=1,2$. They are defined by (5.13) and (5.12).

Note $\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)$ is not yet compact. There are still four types ends, which are,

(I) A pseudo-holomorphic strip escape to the direction $\mathrm{Re}(z)\to+\infty$.
(II) An ASD-connection escape to the direction $\mathrm{Im}(z)\to+\infty$.
(III) An ASD-connection escape to the direction $\mathrm{Re}(z)\to-\infty$.
(IV) An ASD-connection escape to the direction $\mathrm{Im}(z)\to-\infty$.

By definition, ends of type (I) correspond to the union of the following direct products.

$$
\begin{aligned}
&\overset{\circ}{\mathcal{M}}_{k_1',k_2'}((Y,\mathcal{E}_Y);a_-,a_+';R_1,R_2;E_1)\\
&\times\mathcal{M}_{k_1'',k_2''}(R(M_1),R(M_2);\{a_+'\},\{a_+\};E_2).
\end{aligned}
\tag{5.16}
$$

Here $k_1'+k_1''=k_1$, $k_2'+k_2''=k_2$, $E_1+E_2=E$ and $a_+'\in R(M_1)\cap R(M_2)$. (Figure 5.7)

The moduli space $\mathcal{M}_{k_1'',k_2''}(R(M_2),R(M_1);\{a_+'\},\{a_+\};E_2)$ is defined as the union of (4.22). Note in our case we assume $R(M_1)$ is transversal to $R(M_2)$. Therefore each of the connected components of $R(M_1)\cap R(M_2)$ consists of a single point.

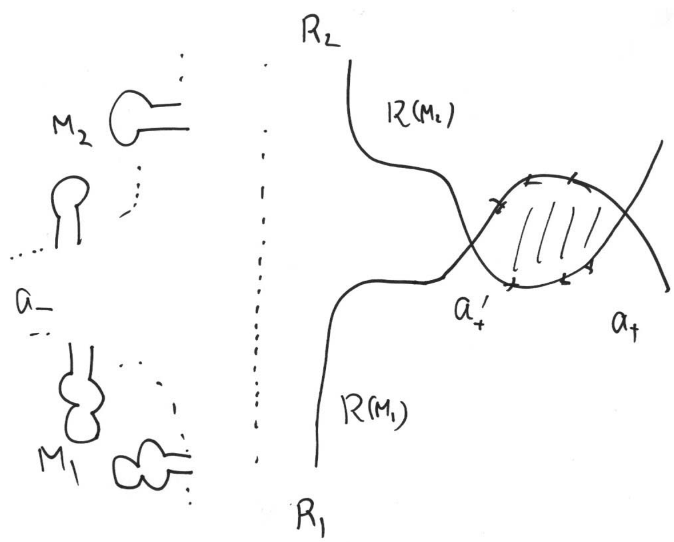

Figure 5.7

By definition, ends of type (III) correspond to the union of the following direct products.

$$\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_+,a'_-,;R_1,R_2;E_1)\times\mathcal{M}(a'_-,a_-;E_2). \tag{5.17}$$

Here $E_1+E_2=E$ and $a'_-\in R(M)=R(M_1)\cap R(M_2)$. The moduli space $\mathcal{M}(a'_-,a_-;E_2)$ is defined as the union of (5.3) (Figure 5.8). Note we need to rotate the bubble appearing at the part $\mathrm{Re}z\to-\infty$ by 90 degree clock-wise direction. Therefore after rotation a'_- will appear in the part $\mathrm{Im}z\to-\infty$. So a'_- and a_- appears as a'_-,a_- in the second factor of (5.17).

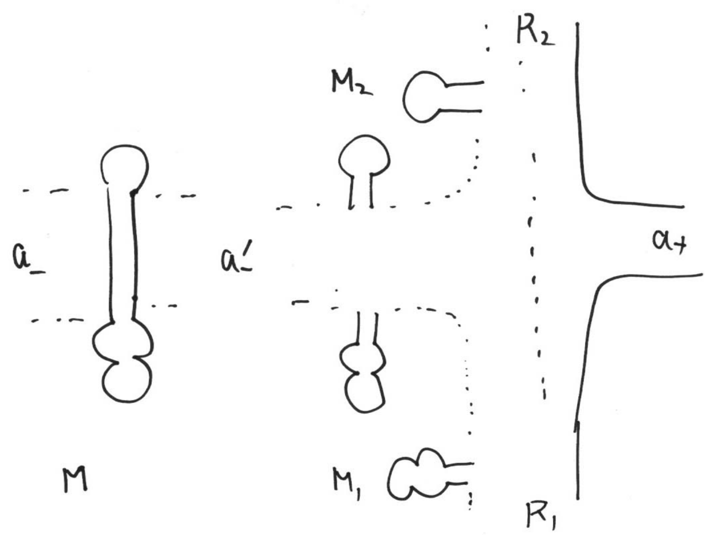

Figure 5.8

We next describe the ends of type (II). For this purpose we use the moduli space $\mathcal{M}_k((M,\mathcal{E}),L;R_-,R_+;E)$ defined in Definition 2.23. This moduli space is defined

as the set of solutions of a partial differential equation on a space which is obtained from a 3-manifold with boundary, which was denoted by M in Section 2. Here we consider either M_1 or $-M_2$ as a 3-manifold M with boundary. So we use the notations $\mathcal{M}_k((M_1,\mathcal{E}_1),L;R_-,R_+;E)$ or $\mathcal{M}_k((-M_2,\mathcal{E}_2),L;R_-,R_+;E)$. (We also take $L = R(M_1)$ or $L = R(M_2)$.)

Now the ends of type (II) is described by the fiber product:

$$\begin{aligned} &\overset{\circ}{\mathcal{M}}_{k_1,k_2'}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2';E_1) \\ &{}_{\mathrm{ev}_2^\infty}\times_{\mathrm{ev}_-} \mathcal{M}_{k_2''}((-M_2,\mathcal{E}_2),R(M_2);R_2',R_2;E_2). \end{aligned} \tag{5.18}$$

Here $k_2 = k_2' + k_2''$, $E = E_1 + E_2$ and R_2' is a connected component of $R(M_2) \times_{R(\Sigma)} R(M_2)$. (Figure 5.9)

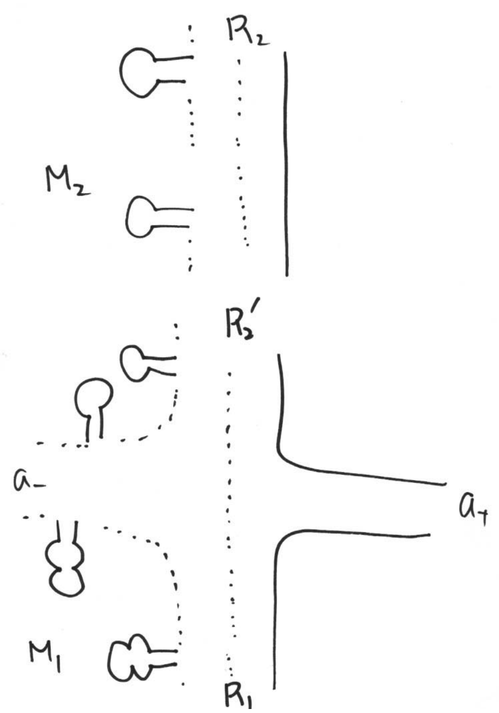

Figure 5.9

Similarly the ends of type (II) is described by the fiber product:

$$\begin{aligned} &\overset{\circ}{\mathcal{M}}_{k_1',k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1',R_2;E_1) \\ &{}_{\mathrm{ev}_1^\infty}\times_{\mathrm{ev}_+} \mathcal{M}_{k_1''}((M_1,\mathcal{E}_1),R(M_1);R_1,R_1';E_2). \end{aligned} \tag{5.19}$$

Here $k_1 = k_1' + k_1''$, $E = E_1 + E_2$ and R_1' is a connected component of $R(M_1) \times_{R(\Sigma)} R(M_1)$. (Figure 5.10)

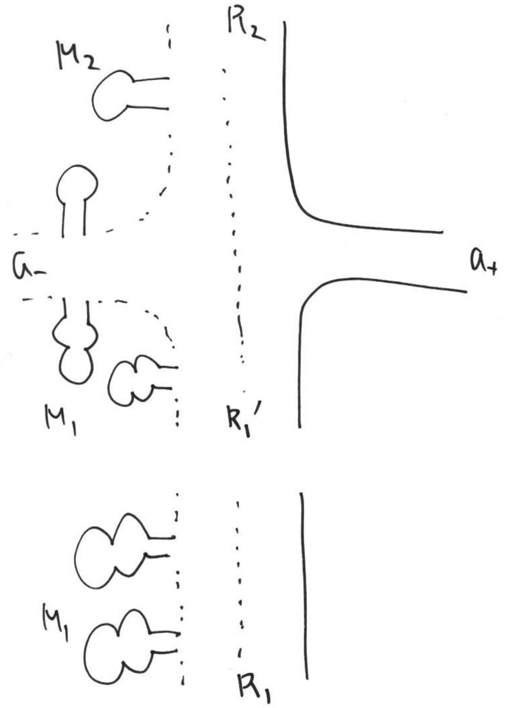

Figure 5.10

Proposition 5.9. *We can compactify* $\overset{\circ}{\mathcal{M}}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)$ *to a compact Hausdorff space* $\mathcal{M}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)$*. It carries a virtual fundamental chain, whose boundary is the sum of the virtual fundamental chains of the following 6 types of spaces.*

(1)
$$\mathcal{M}_{k_1',k_2'}((Y,\mathcal{E}_Y);a_-,a_+';R_1,R_2;E_1)\times\mathcal{M}_{k_1'',k_2''}(R(M_1),R(M_2);\{a_+'\},\{a_+\};E_2),$$
where the notations are as in (5.16).

(2)
$$\mathcal{M}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-',a_+;R_1,R_2;E_1)\times\mathcal{M}(a_-,a_-';E_2),$$
where the notations are as in (5.17).

(3)
$$\mathcal{M}_{k_1,k_2'}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2';E_1)_{\mathrm{ev}_2^\infty}\times_{\mathrm{ev}_-}\mathcal{M}_{k_2''}((-M_2,\mathcal{E}_2),R(M_2);R_2',R_2;E_2),$$
where the notations are as in (5.18).

(4)
$$\mathcal{M}_{k_1',k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1',R_2;E_1)_{\mathrm{ev}_1^\infty}\times_{\mathrm{ev}_+}\mathcal{M}_{k_1''}((M_1,\mathcal{E}_1),R(M_1);R_1,R_1';E_2),$$
where the notations are as in (5.19).

(5)
$$\mathcal{M}_{k_1',k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E_1)_{\mathrm{ev}_i^{(1)}}\times_{\mathrm{ev}_0}\mathcal{M}_{k_1''}(R(M_1);E_2),$$
where $k_1+1=k_1'+k_1''$*,* $E=E_1+E_2$*,* $i=1,\dots,k_1'$*.*

(6)

$$\mathcal{M}_{k_1,k_2'}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E_1)\,{}_{\mathrm{ev}_i^{(2)}}\times_{\mathrm{ev}_0}\mathcal{M}_{k_2''}(R(M_2);E_2),$$

where $k_2+1=k_2'+k_2''$, $E=E_1+E_2$, $i=1,\dots,k_2'$.

Proof. The first 4 items describe the boundaries (I), (II), (III), (IV), respectively. (5) describes the disk bubble on $\partial_4 W$ and (6) describes the disk bubble on $\partial_1 W$. All the other bubbles occur in codimension 2 or higher. $\square$

We next rewrite Proposition 5.9 to an algebraic formula.

We first need a digression. The second factor in Proposition 5.9 (3). we used $-M_2$ and we use ev_- to take fibre product. In the construction of right filtered A_∞ module structure in Section 2 the evaluation map ev_- corresponds to the input variables. When we change from $-M_2$ to M_2 the input variables becomes the output variables. Namely we have an isomorphism

$$\mathcal{M}_k((-M_2,\mathcal{E}_2),R(M_2);R_-,R_+;E)\cong\mathcal{M}_k((M_2,\mathcal{E}_2),R(M_2);R_+,R_-;E)$$

which intertwine $(\mathrm{ev}_-,\mathrm{ev}_+)$ to $(\mathrm{ev}_+,\mathrm{ev}_-)$. Moreover i-th evaluation map ev_i : $\mathcal{M}_k((-M_2,\mathcal{E}_2),R(M_2);R_-,R_+;E)\to R(M_2)$ becomes $(k-i)$-th evaluation map $\mathrm{ev}_{k-i}:\mathcal{M}_k((M_2,\mathcal{E}_2),R(M_2);R_-,R_+;E)\to R(M_2)$ by this isomorphism.

This fact is used in Lemma 5.10 below.

We now define the map

$$\begin{aligned}\Phi_{k_1,k_2;E}:&CF(M_1,R(M_1))\otimes CF(R(M_1))^{k_2\otimes}\\&\otimes CF(R(M_1),R(M_2))\\&\otimes CF(M_2,R(M_2))\otimes CF(R(M_2))^{k_1\otimes}\to CF(M;\mathcal{E})\end{aligned}\tag{5.20}$$

by the formula

$$\begin{aligned}&\Phi_{k_1,k_2;E}(y_1;x^{(1)}_1,\dots,x^{(1)}_{k_1};a_+;y_2;x^{(2)}_1,\dots,x^{(2)}_{k_2})\\&=\sum_{a_-}\#\big(\mathcal{M}_{k_1,k_2}((Y,\mathcal{E}_Y);a_-,a_+;R_1,R_2;E)\\&\qquad{}_{\mathrm{ev}}\times(y_1\times x^{(1)}_1\times\cdots\times x^{(1)}_{k_1}\times y_2\times x^{(2)}_1\times\cdots\times x^{(2)}_{k_2})\big)[a_-].\end{aligned}\tag{5.21}$$

Here $y_1\in CF(M_1,R(M_1))$ and $y_2\in CF(M_2,R(M_2))$, $a_+\in CF(R(M_1),R(M_2))$, $a_-\in CF(M;\mathcal{E})$.

We then put

$$\Phi_{k_1,k_2}=\sum_E T^E\Phi_{k_1,k_2;E}.\tag{5.22}$$

Proposition 5.9 implies the next formula:

Lemma 5.10.

$$\begin{aligned}&\partial(\Phi_{k_1,k_2}(y_1;x^{(1)}_1,\dots,x^{(1)}_{k_1};a_+;y_2;x^{(2)}_1,\dots,x^{(2)}_{k_2}))\\&=\sum\Phi_{k_1'',k_2''}(y_1;x^{(1)}_1,\dots,x^{(1)}_{k_1''};\mathfrak{n}_{k_1',k_2'}(x^{(1)}_{k_1''+1},\dots,x^{(1)}_{k_1}a_+;x^{(2)}_1,\dots,x^{(2)}_{k_2'});y_2;x^{(2)}_{k_2'+1},\dots,x^{(2)}_{k_2})\\&\quad+\sum\Phi_{k_1'',k_2}(\mathfrak{n}_{k_1'}(y_1;x^{(1)}_1,\dots,x^{(1)}_{k_1'});\dots,x^{(1)}_{k_1};a_+;y_2;x^{(2)}_1,\dots,x^{(2)}_{k_2})\\&\quad+\sum\Phi_{k_1,k_2''}(y_1;x^{(1)}_1,\dots,x^{(1)}_{k_1};a_+;\mathfrak{n}_{k_2'}(y_2;x^{(2)}_{k_2},\dots,x^{(2)}_{k_2''+1});x^{(2)}_1,\dots,x^{(2)}_{k_2''})\\&\quad+\sum\Phi_{k_1'',k_2}(y_1;x^{(1)}_1,\dots,\mathfrak{m}_{k_1'}(x^{(1)}_i,\dots,x^{(1)}_{i+k_1'-1}),\dots,x^{(1)}_{k_1};a_+;y_2;x^{(2)}_1,\dots,x^{(2)}_{k_2})\\&\quad+\sum\Phi_{k_1,k_2''}(y_1;x^{(1)}_1,\dots,x^{(1)}_{k_1};a_+;y_2;x^{(2)}_1,\dots,\mathfrak{m}_{k_2'}(x^{(2)}_i,\dots,x^{(2)}_{i+k_2'-1}),\dots,x^{(2)}_{k_2}).\end{aligned}$$

Here ∂ in the first line is Floer's boundary operator (Definition 5.3).

Proof. 1st, 2nd, 3rd, 4th, 5th, 6th lines of the formal corresponds to items (2), (1), (4), (3), (5), (6) of Proposition 5.9, respectively. □

We define a map $\Phi : CF(R(M_1), R(M_2)) \to CF(M; \mathcal{E})$ by

$$\Phi(a_+) = \sum_{k_1,k_2} \Phi_{k_1,k_2}(\mathbf{1}_{M_1}; b_{M_1}, \dots, b_{M_1}; a_+; \mathbf{1}_{M_2}; b_{M_2}, \dots, b_{M_2}). \tag{5.23}$$

Lemma 5.11. Φ *is a chain map. Namely*

$$\partial \circ \Phi = \Phi \circ {}^{b_{M_1}}(\mathfrak{n}_0)^{b_{M_2}}$$

Proof. We recall

$${}^{b_{M_1}}(\mathfrak{n}_0)^{b_{M_2}}(a_+) = \sum_{k_1,k_2} \mathfrak{n}_{k_1,k_2}(b_{M_1}, \dots, b_{M_1}; a_+; b_{M_2}, \dots, b_{M_2}).$$

We put $y_1 = \mathbf{1}_{M_1}$, $y_2 = \mathbf{1}_{M_2}$, $x_i^{(1)} = b_{M_1}$, $x_i^{(2)} = b_{M_2}$ in Lemma 5.10. Then the first line becomes $\partial \circ \Phi$. The second line becomes ${}^{\mathfrak{b}_{M_1}}(\mathfrak{n}_0)^{\mathfrak{b}_{M_2}}(a_+)$. The third line cancels since $d^{b_{M_2}}(\mathbf{1}_{M_1}) = 0$. The fourth line cancels since $d^{b_{M_1}}(\mathbf{1}_{M_2}) = 0$. The fifth line cancels since b_{M_1} is a bounding cochain. The sixth line cancels since b_{M_2} is a bounding cochain. □

We thus defined a chain map (5.1). To show that it is an isomorphism it suffices to prove the next:

Lemma 5.12. $\Phi \equiv \mathrm{id} \mod \Lambda_+^{\mathbb{Z}_2}$.

Proof. We remark that both $CF(R(M_1), R(M_2))$ and $CF(M; \mathcal{E})$ are free $\Lambda_0^{\mathbb{Z}_2}$ module with basis $R(M) = R(M_1) \cap R(M_2)$. So the statement makes sense.

Since $b_{M_1}, b_{M_2} \equiv 0 \mod \Lambda_+^{\mathbb{Z}_2}$ it suffices to study $\Phi_{0,0;0}$.

The map $\Phi_{0,0;0}$ is defined by the moduli space $\mathcal{M}_{0,0}((Y, \mathcal{E}_Y); a_-, a_+; R_1, R_2; 0)$.

From the definition it consists of equivalence classes of elements $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u)$ such that $\mathfrak{A}$ is a flat connection u is a constant map and $\Omega = W$. Therefore $\mathfrak{A} = a_- = a_+$. Using the fact that $\mathbf{1}_{M_1}$ and $\mathbf{1}_{M_2}$ are fundamental classes also we can prove the lemma easily. □

The proof of Theorem 1.1 and other results stated in the introduction are now complete modulo the construction and the check of its basic properties of the moduli spaces we used. □

Remark 5.13. This remark is a continuation of Remark 4.15. Let M_1 and M_2 be symplectic manifolds and L_{12} an immersed Lagrangian submanifold of $M_1 \times -M_2$. Let L_1, L_1' be immersed Lagrangian submanifolds of M_1. We obtain an immersed Lagrangian submanifolds L_2 (resp. L_2') of M_2 by using L_1 and L_{12} (resp. L_1' and L_{12}). We assume M_1, M_2, L_{12}, L_1, L_1' are spin. Then L_2 and L_2' are spin. Let b_1, b_1', b_{12} be bounding cochains of the filtered A_∞ algebra associated to L_1, L_1', L_{12}, respectively.

Then by Theorem 3.13, we obtain bounding cochains b_2 and b_2' of L_2 and L_2', respectively. In the same way as the construction of the chain map (5.1) and the proof of Theorem 1.1 (2) we can construct a canonical homomorphism

$$\Phi_{(L_{12},b_{12})} : HF((L_1, b_1), (L_1', b_1')) \to HF((L_2, b_2), (L_2', b_2')). \tag{5.24}$$

The construction is based on the following Figure 5.11.

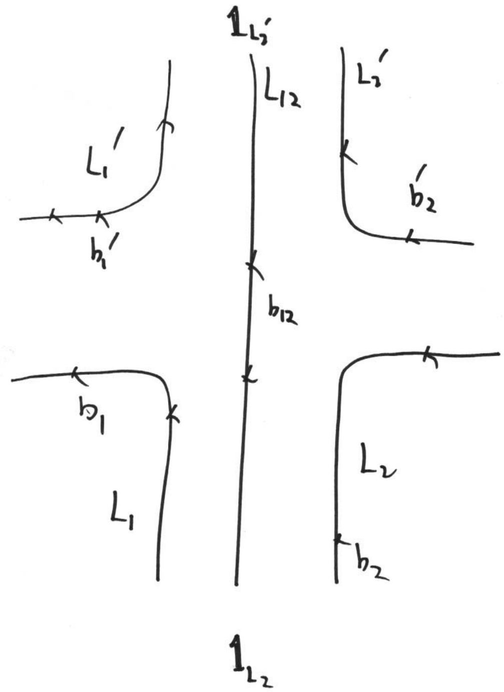

Figure 5.11

We can enhance the homomorphism $\Phi_{(L_{12},b_{12})}$ to an A_∞ functor

$$\Phi_{(L_{12},b_{12})} : \mathscr{FUK}(M_1) \to \mathscr{FUK}(M_2). \tag{5.25}$$

Note this map $\Phi_{(L_{12},b_{12})}$ may not be an isomorphism in general. In fact $L_1 \cap L_1' \neq L_2 \cap L_2'$ in general.

This functor was constructed by Ma'u-Wehrheim-Woodward [MWW] under certain additional assumptions. In the same way as Lekili-Lipyanskiy [LL] we can prove its compatibility with composition. Namely:

Theorem 5.14. *Let M_3 be another spin symplectic manifold and L_{23} an immersed spin Lagrangian submanifold of $M_2 \times -M_3$. Suppose b_{23} is a bounding cochain of L_{23}. Let $\tilde L_{13} = \tilde L_{12} \times_{M_2} \tilde L_{23}$. (Then $i_{L_{13}} : \tilde L_{13} \to L_{13} \subset M_1 \times M_3$ is an immersed Lagrangian submanifold.)*

Applying Lagrangian correspondence L_{23} to L_2 and L_2' we obtain immersed Lagrangian submanifolds L_3 and L_3' respectively. (They coincide with the immersed Lagrangian submanifolds obtained from L_1 and L_1' by L_{13}.)

We apply Theorem 3.13 to (L_2, b_2) (resp. (L_2', b_2')) and (L_{23}, b_{23}) and obtain bounding cochain b_3 (resp. b_3') of L_3 (resp. L_3').

1. *The immersed Lagrangian submanifold L_{13} is unobstracted. Moreover the bounding cochains b_{12} and b_{23} induce a bounding cochain b_{13} of L_{13} in a canonical way, up to gauge equivalence.*
2. *The bounding cochain b_3 (resp. b_3') is gauge equivalent to the bounding cochains obtained by applying Theorem 3.13 to (L_{13}, b_{13}) and b_3 (resp. b_3').*
3. *We have equality*

$$\Phi_{(L_{23},b_{23})} \circ \Phi_{(L_{12},b_{12})} = \Phi_{(L_{13},b_{13})} : HF((L_1,b_1),(L_1',b_1')) \to HF((L_3,b_3),(L_3',b_3')).$$

Sketch of the proof. Proof of (1): We consider the maps u from the cylinder of infinite length in the Figure 5.12 below, where three regions W_1, W_2 and W_3 are mapped to M_1, M_2 and M_3, respectively, by a pseudo-holomorphic map u. We also require that the three lines C_{12}, C_{23} and C_{13} are mapped to L_{12}, L_{23} and L_{13} respectively.

We are in Bott-Morse situation and the asymptotic limit as we go to plus or minus infinity is an element of

$$L_{13} \times_{M_1 \times M_3} L_{13} = (L_{12} \times L_{23} \times L_{13}) \times_{(M_1 \times M_2 \times M_3)^2} \Delta_{M_1 \times M_2 \times M_3}$$

Here $\Delta_{M_1 \times M_2 \times M_3}$ is a diagonal. (We can work out analytic detail by regarding this moduli space as those to define Floer homology

$$HF(L_{12} \times L_{23} \times L_{13}, \Delta_{M_1 \times M_2 \times M_3})$$

of a pair of immersed Lagrangian submanifolds of $(M_1 \times M_2 \times M_3)^2$.)

We use bounding cochains b_{12}, b_{23} to cancel the effect of bubbles on C_{12}, C_{23}, respectively, in the same way as [FOOO1].

Then using marked points on the line C_{13}, we can define a structure of filtered right A_∞ module on $CF(L_{12} \times L_{23} \times L_{13}, \Delta_{M_1 \times M_2 \times M_3})$ over the filtered A_∞ algebra $CF(L_{13})$. In the same way as Section 3, we can show that the fundamental class is a cyclic element of this right filtered A_∞ module. Thus applying Proposition 3.5, we obtain b_{13}.

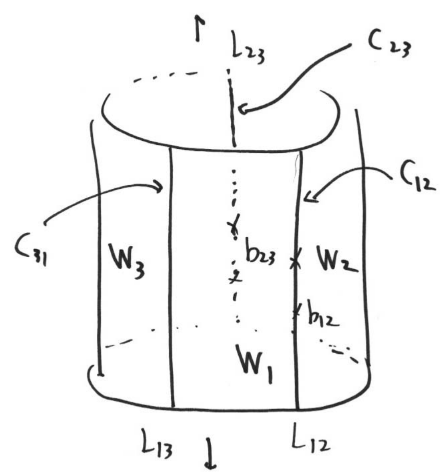

Figure 5.12

Proof of (2): We denote by b_3^1 (resp. $b_3'^1$) be the bounding cochain on L_3 (resp. on L_3') obtained from (L_{23}, b_{23}) and (L_2, b_2) (resp. (L_2', b_2'))). We remark that (L_2, b_2) (resp. (L_2', b_2')) is obtained from (L_{12}, b_{12}) and (L_1, b_1) (resp. (L_1', b_1')).

We denote by b_3^2 (resp. $b_3'^2$) the bounding cochain on L_3 (resp. on L_3') obtained from (L_{13}, b_{13}) and (L_1, b_1) (resp. (L_1', b_1'))).

Let (L, b) be a pair of an immersed spin Lagrangian submanifold of M_3 and its bounding cochain. We consider the moduli space of pseudo-holomorphic quilt as in Figure 5.13 below:

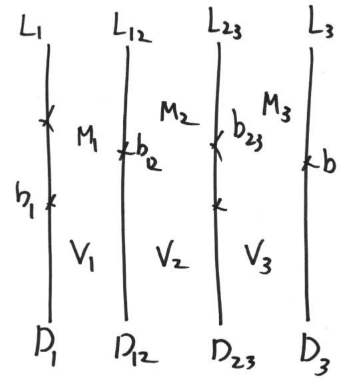

Figure 5.13

Here the domains V_1, V_2 and V_3 are mapped to M_1, M_2 and M_3 respectively and the maps are pseudo-holomorphic. We also require the four lines D_1, D_{12}, D_{23} and D_3 are mapped to L_1, L_{12}, L_{23}, L respectively. Using b_1, b_{12}, b_{23} and b to cancel the bubble on the lines D_1, D_{12}, D_{23} and D_3, respectively, we obtain a chain complex, on the $\Lambda_0^{\mathbb{Q}}$ module.

$$C(L_1 \times_{M_1} L_{12} \times_{M_2} L_{23} \times_{M_3} L; \Lambda_0^{\mathbb{Q}}). = CF(L_1; L_{12}; L_{23}; L)$$

We write its cohomology by $HF((L_1, b_1); (L_{12}, b_{12}); (L_{23}, b_{23}); (L, b))$. We claim

$$HF((L_1, b_1); (L_{12}, b_{12}); (L_{23}, b_{23}); (L, b)) \cong HF((L_1, b_1) \times (L, b); (L_{13}, b_{13})). \quad (5.26)$$

The proof of (5.26) is basically the same as the proof of the corresponding result in [LL]. Namely we use the next Figure 5.14. (Figure 5.14 is the same as [LL, Figure 1].)

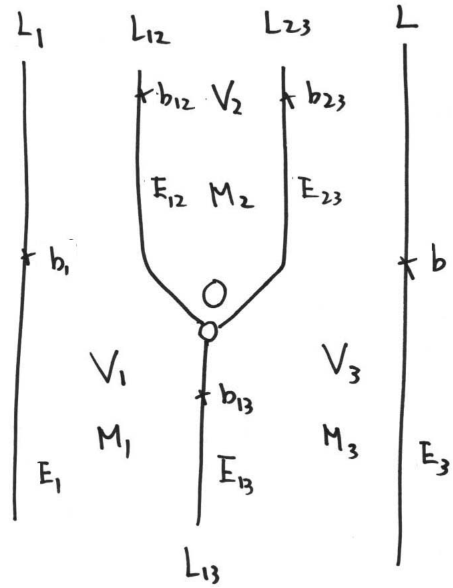

Figure 5.14

We consider maps u from V in Figure 5.14 such that V_1, V_2, V_3 are mapped by u to M_1, M_2, M_3, respectively. We also require that the curves E_1, E_{12}, E_{23}, E_{13}, E_3 are mapped to L_1, L_{12}, L_{23}, L_{13}, L, respectively. Moreover we conformally identify a neighborhood of the point O as the (half of the) cylinder as in Figure 5.12.

We use bounding cochains b_1, b_{12}, b_{23}, b_{13}, b to cancel bubbles on the lines E_1, E_{12}, E_{23}, E_{13}, E_3, respectively, in the same way as [FOOO1]. At the point O we use the cyclic element (the fundamental chain) to define asymptotic boundary condition.

We remark that V_1, V_2, V_3 are in the clock-wise order in Figure 5.14 and are in counter-clock-wise order in Figure 5.12. This changes the cycle we put to O from output variable to input variable.

Then in the same way as the proof of Theorem 1.1 (2) given in this section, the remaining boundary component is the next two cases.

(I) A pseudo-holomorphic strip escape to the direction $\mathrm{Im} z \to +\infty$.
(II) A pseudo-holomorphic strip escape to the direction $\mathrm{Im} z \to -\infty$.

(I) gives the boundary operator defining $HF((L_1, b_1); (L_{12}, b_{12}); (L_{23}, b_{23}); (L, b))$. (II) given the boundary operator defining $HF((L_1, b_1) \times (L, b); (L_{13}, b_{13}))$. Therefore we obtain a chain map between two chain complexes defining them. We can show that this chain map is congruent to the identity map mod $\Lambda_0^{\mathbb{Q}}$, since the energy zero element of this moduli space is a constant map. This proves (5.26).

We next prove:

$$HF((L_1, b_1); (L_{12}, b_{12}); (L_{23}, b_{23}); (L, b)) \cong HF((L_3, b_3^1), (L, b)). \tag{5.27}$$

To prove (5.27) we consider the next Figure 5.15. We consider a map u such that it is a map to M_1, M_2, M_3 on the domain U_1, U_2, U_3, respectively. We also require u maps F_1, F_2, F_3, F_3', F_{12}, F_{23} to L_1, L_2, L_3, L, L_{12}, L_{23} respectively.

We put bounding cochains b_1, b_2, b_3^1, b, b_{12}, b_{23} on F_1, F_2, F_3, F_3', F_{12}, F_{23}, respectively to cancel the contribution of disk bubbles there.

We regard the ends O_{12} and O_{23} are as Figure 4.15 and put asymptotic boundary conditions by using fundamental classes there.

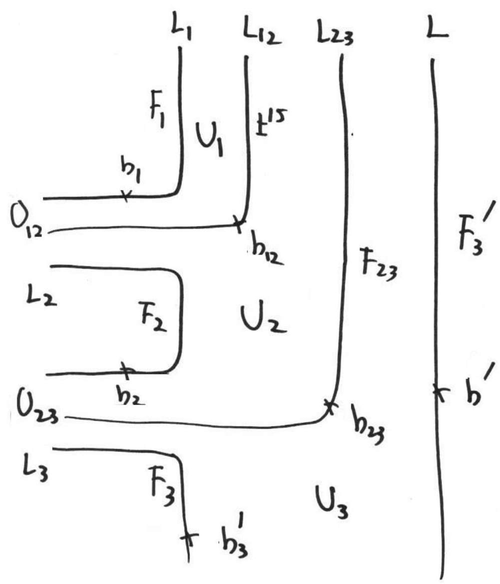

Figure 5.15

Thus the remaining boundary component of this moduli space is as in (I) and (II) above. (I) corresponds to the boundary operator defining left hand side of (5.27). (II) corresponds to the boundary operator defining right hand side of (5.27). Therefore we obtain a chain map. Using the fact that energy zero solution corresponds to the constant map and we put fundamental classes at O_{12} and O_{23}, we find that this chain map is congruent to the identity map modulo $\Lambda_0^{\mathbb{Q}}$. We proved (5.27).

On the other hand, Theorem 3.13 implies

$$HF((L_1,b_1)\times(L,b),(L_{13},b_{13}))\cong HF((L_3,b_3^2),(L,b)). \tag{5.28}$$

Combining (5.26), (5.27), (5.28) we find

$$HF((L_3,b_3^1),(L,b))\cong HF((L_3,b_3^2),(L,b)). \tag{5.29}$$

It is straightforward to check that the isomorphism (5.29) is functorial. Namely two filtered A_∞ functors represented by (L_3,b_3^1) and by (L_3,b_3^2) are homotopy equivalent. Using A_∞-Yoneda's lemma two objects (L_3,b_3^1) and (L_3,b_3^2) are homotopy equivalent. It implies (2).

We omit the proof of (3). (We will prove it in [Fu9].) We can also enhance the isomorphism in (3) to a homotopy equivalence between two filtered A_∞ functors. □

6. Concluding remarks

6.1. **Topological field theory description.** We remark that the results of this paper provides 2-3 dimensional topological field theory picture of Gauge theory

Floer homology. (Such a picture is initiated by [D4], [Fu2] and also by G. Segal, in early 1990's.)

Let $(M, \mathcal{E}_M)$ be as in Situation 2.1. We divide ∂M into two pieces, input part $\partial_{\text{in}} M = \Sigma_{\text{in}}$ and output part $\partial_{\text{in}} M = \Sigma_{\text{out}}$. For $\partial_{\text{out}} M = \Sigma_{\text{out}}$ we invert the orientation.

Definition 6.1. We call this situation that $(M, \mathcal{E}_M)$ is a *cobordism* from $(\Sigma_{\text{in}}, \mathcal{E}_{\text{in}})$ to $(\Sigma_{\text{out}}, \mathcal{E}_{\text{out}})$.

The space $R(\Sigma)$ with it symplectic structure ω_Σ is written as

$$(R(\Sigma), \omega_\Sigma) = (R(\Sigma_{\text{in}}), \omega_{\Sigma_{\text{in}}}) \times (R(\Sigma_{\text{out}}), -\omega_{\Sigma_{\text{out}}}).$$

The space of flat connections $R(M)$ of M is an immersed Lagrangian submanifold of it. By Theorem 1.1 (1) we obtain a bounding cochain b_M of the filtered A_∞ algebra associated to $R(M)$. Therefore (5.25) induces a filtered A_∞ functor

$$\Phi_M : \mathscr{FUK}(R(\Sigma_{\text{in}})) \to \mathscr{FUK}(R(\Sigma_{\text{out}})). \tag{6.1}$$

This construction behave functorially as follows. Let $(M_{12}, \mathcal{E}_{12})$ (resp. $(M_{23}, \mathcal{E}_{23})$) be a cobordism from $(\Sigma_1, \mathcal{E}_1)$ to $(\Sigma_2, \mathcal{E}_2)$ (resp. from $(\Sigma_2, \mathcal{E}_2)$ to $(\Sigma_3, \mathcal{E}_3)$).

We glue $(M_{12}, \mathcal{E}_{12})$ and $(M_{23}, \mathcal{E}_{23})$ along $(\Sigma_2, \mathcal{E}_2)$ to obtain $(M_{13}, \mathcal{E}_{13})$.

Theorem 6.2. *The composition $\Phi_{M_{23}} \circ \Phi_{M_{12}}$ is homotopy equivalent to $\Phi_{M_{13}}$ as filtered A_∞ functors.*

Sketch of the proof. We remark that in case $\Sigma_1 = \Sigma_3 = \emptyset$, this is Theorem 1.1 (2). For the proof of the general case we first consider filtered A_∞ bifunctor

$$\mathscr{FUK}(R(\Sigma_1))^{\text{op}} \times \mathscr{FUK}(R(\Sigma_2)) \to \mathscr{CH}. \tag{6.2}$$

Here op stands for the opposite A_∞ category. (See [Fu6, Definition 7.8].) Note the left hand side of (6.2) is a full subcategory of $\mathscr{FUK}(-R(\Sigma_1) \times R(\Sigma_2))$. (See [Ln].)

Therefore object of $\mathscr{FUK}(R(\Sigma_1) \times -R(\Sigma_2))$ represents a functor as in (6.2). The pair $(R(M_{12}), b_{M_{12}})$ is such an object and it represents

$$((L_1, b_1), (L_2', b_2')) \mapsto HF(\Phi_{M_{12}}(L_1, b_1), (L_2', b_2')). \tag{6.3}$$

This is a consequence of Theorem 3.13. Namely (6.3) is isomorphic to

$$HF((R(M_{12}), b_{M_{12}}), (L_1, b_1) \times (L_2', b_2')) \tag{6.4}$$

by Theorem 3.13.

On the other hand, Theorem 1.4 implies that the Floer homology group (6.4) is isomorphic to $HF((M_{12}, \mathcal{E}_{12}), (L_1, b_1) \times (L_2', b_2')))$. We can rephrase the definition of this group in Section 2 and find that it is a homology group of the chain complex $C(R(M_{12}) \times_{R(\Sigma)} (L_1 \times L_2'); \Lambda_0^{\mathbb{Z}_2})$ whose boundary operator is defined by using the moduli space of solutions of (2.12), (2.14) on the domain described in the next figure.

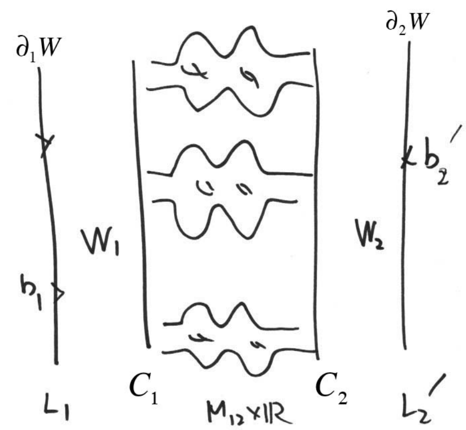

Figure 6.1

Here the domain is divided into 3 parts. On two of them W_1 and W_2 we use degenerate metrics $0g_{\Sigma_1} + ds^2 + dt^2$ and $0g_{\Sigma_2} + ds^2 + dt^2$, respectively and our equation is a holomorphic curve equation to $R(\Sigma_1)$ and to $R(\Sigma_2)$, respectively. The third part is $M_{12} \times \mathbb{R}$, on which we consider the ASD-equation (2.12). The metric is of the form $\chi^2 g_{\Sigma_2} + ds^2 + dt^2$ near their borderline curves C_1 and C_2, where χ is a function similar to those we used several times.

We require that the holomorphic curve takes boundary value in L_1 and L_2' on the boundary $\partial_1 W$ and $\partial_2 W$, respectively.

We require asymptotic boundary conditions for $t \to \pm\infty$ by using elements of $L_1 \times_{R(\Sigma_1)} R(M_{12}) \times_{R(\Sigma_2)} L_2'$. We cancel the contribution of the disk bubble on $\partial_1 W$ and on $\partial_2 W$ by using bounding cochains b_1 and b_2' in the same way as [FOOO1]. We thus obtain a boundary operator on $C(R(M_{12}) \times_{R(\Sigma)} (L_1 \times L_2'); \Lambda_0^{\mathbb{Z}_2})$.

It is easy to see from definition that the homology group of this chain complex is

$$HF((M_{12}, \mathcal{E}_{12}), ((L_1 \times L_2'), (b_1 \times b_2'))). \tag{6.5}$$

The isomorphism (6.4) $\cong$ (6.5) is a consequence of Theorem 1.4.

We go back to the proof of Theorem 6.2. Let (L_1, b_1) (resp. (L_3', b_3')) be an object of $\mathscr{FUK}(R(\Sigma_1))$ (resp. $\mathscr{FUK}(R(\Sigma_3))$). We put $\Phi_{M_{12}}(L_1, b_1) = (L_2, b_2)$. To prove Theorem 6.2 it suffices to construct a functorial isomorphism:

$$HF(\Phi_{M_{23}}(L_2, b_2), (L_3', b_3')) \cong HF(\Phi_{M_{13}}(L_1, b_1), (L_3', b_3')). \tag{6.6}$$

We use the moduli space obtained by the solution of (2.12), (2.14) on the domain described in the next figure.

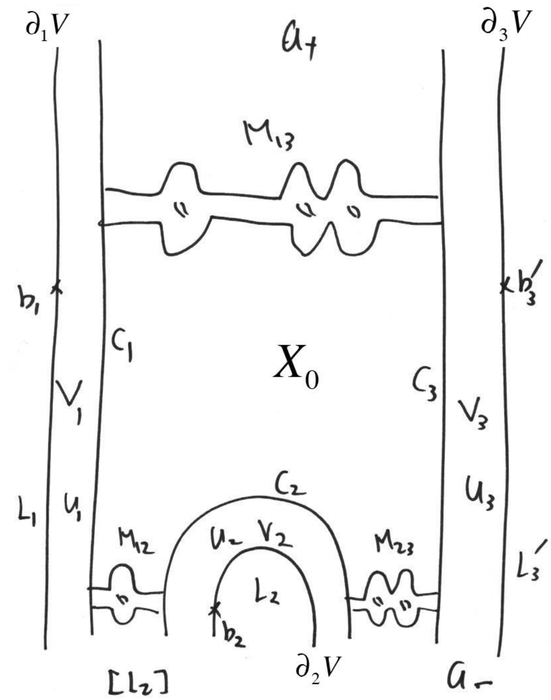

Figure 6.2

This domain is divided into 4 pieces. On three of them V_1, V_2, V_3 we use the degenerate metrics $0g_{\Sigma_1} + ds^2 + dt^2$, $0g_{\Sigma_2} + ds^2 + dt^2$, $0g_{\Sigma_3} + ds^2 + dt^2$, respectively. So our equation is a holomorphic curve equation to $R(\Sigma_1)$, $R(\Sigma_2)$, $R(\Sigma_3)$ on V_1, V_2, V_3 respectively.

The fourth part of the domain X_0 is a Riemannian 4-manifold. It has three ends. Two of them are in the part $t \to -\infty$ and are isometric to $M_{12} \times (-\infty, -c]$ and to $M_{23} \times (-\infty, -c]$. The third end is in the part $t \to +\infty$ and is isometric to $M_{13} \times (c, +\infty)$. On X_0 we require the ASD-equation.

X_0 intersects with $\Sigma_i \times V_i$ on $\Sigma \times C_i$. Near the borderline curve C_i (which is diffeomorphic to $\mathbb{R}$) the 'metric' is isometric to one of the form $\chi(s)^2 g_{\Sigma_1} + ds^2 + dt^2$ where χ is a similar function as we used several times in this paper.

We put $\partial_i V = \partial V_i \setminus C_i$. They are diffeomorphic to $\mathbb{R}$. On $\partial_i V$ we put the following boundary conditions. We put $\Phi_{12}(L_1, b_1) = (L_2, b_2)$.

(1) The image of the restriction of u_1 to $\partial_1 V$ is in L_1.
(2) The image of the restriction of u_2 to $\partial_2 V$ is in L_2.
(3) The image of the restriction of u_3 to $\partial_3 V$ is in L_3'.

Actually in case L_1, L_2 or L_3' are immersed we need to state the boundary condition a bit more carefully. Since we explained the way how to do so in a similar situation several times already in this paper we omit it here.

We now consider the asymptotic boundary conditions for three ends. We take

$$a_-, a_+ \in \tilde{L}_1 \times_{R(M_{13})} \tilde{L}_3' \cong \tilde{L}_2 \times_{R(M_{23})} \tilde{L}_3'. \tag{6.7}$$

Note the isomorphism in (6.7) can be proved by

$$R(M_{13}) = R(M_{12}) \times_{R(\Sigma_2)} R(M_{23})$$

and

$$\tilde{L}_2 = \tilde{L}_1 \times_{R(\Sigma_1)} R(M_{12}).$$

(I) At the end isometric to $M_{12} \times (-\infty, -c]$ we require that our connection converges to a flat connection in L_2. In other words we put the fundamental class

$$[L_2] \in C(M_{12} \times_{R(\Sigma_1)\times R(\Sigma_2)} (\tilde{L}_1 \times \tilde{L}_2)) \cong C(\tilde{L}_2 \times_{R(\Sigma_2)} \tilde{L}_2).$$

(II) At the end isometric to $M_{23} \times (-\infty, -c]$ we require that our connection is asymptotic to a_-.

(III) At the end isometric to $M_{13} \times [c, +\infty)$ we require that our connection is asymptotic to a_+.

Finally we use bounding cochains b_1, b_2 and b_3^1 on $\partial_1 V$, $\partial_2 V$ and $\partial_3 V$, respectively, to cancel the contribution of the disk bubble there.

We use this moduli space in case when its virtual dimension is 0 to define a matrix element $\langle \Phi(a_-), a_+\rangle$ of the map

$$\Phi : CF(\Phi_{M_{23}}(L_2, b_2), (L'_3, b'_3)) \to CF((M_3, \mathcal{E}_3), (L_1, b_1) \times (L'_3, b'_3)). \tag{6.8}$$

We claim that Φ is a chain map. To prove it we consider the same moduli space when its virtual dimension is 1. We study its boundary.

The contribution of the codimension one boundary corresponding to the disk bubbles on $\partial_i V$ is zero since we used bounding cochains b_1, b_2 and b_3^1 to cancel it. The contribution of the codimension one boundary corresponds to the end of type (I) also vanishes. This is because of the choice of b_2. (In fact we used Proposition 3.5 to find b_2. In other words, the fundamental class which we put as an asymptotic value at this ends, is a cycle.)

The two other ends (II) and (III) correspond to the boundary operator of the source and target of (6.8), respectively. Thus (6.8) is a chain map.

The energy 0 solution of our equation consists of flat connections and constant maps. We use this fact to prove that (6.8) is congruent to the identity map modulo $\Lambda_0^{\mathbb{Z}_2}$.

We thus constructed an isomorphism (6.6).

We omit the proof of functoriality. $\square$

Theorem 6.3. *We have the following additional properties of our functor $M_{12} \mapsto \Phi_{M_{12}}$ and $\Sigma \mapsto \mathscr{FUK}(R(\Sigma))$.*

(1) $\mathscr{FUK}(R(\Sigma_1 \sqcup \Sigma_2)) = \mathscr{FUK}(R(\Sigma_1) \times \mathscr{FUK}(R(\Sigma_2))$.

(2) $\mathscr{FUK}(R(-\Sigma)) = \mathscr{FUK}(R(\Sigma))^{\mathrm{op}}$. *Here* op *denotes the opposite filtered A_∞ category.*

(3) *We invert the orientation of M_{12} and obtain M_{21}. We use the same bundle $\mathcal{E}_{12} = \mathcal{E}_{21}$. Then $\Phi_{M_{12}}$ is the adjoint functor to $\Phi_{M_{21}}$. Namely there exists an isomorphism*

$$HF(\Phi_{M_{12}}(L_1, b_1), (L'_2, b'_2)) \cong HF(\Phi_{M_{21}}(L'_2, b'_2), (L_1, b_1)),$$

which is functorial. Namely the left and right hand sides are homotpy equivalent as filtered A_∞ bifunctors (6.2).

(4) *Suppose $\partial_{\mathrm{out}}(M, \mathcal{E}) \cong -\partial_{\mathrm{in}}(M, \mathcal{E})$. We glue $\partial_{\mathrm{out}}(M, \mathcal{E})$ and $\partial_{\mathrm{in}}(M, \mathcal{E})$ in $(M, \mathcal{E})$ to obtain a closed manifold with bundle, $(\hat{M}, \hat{\mathcal{E}})$. Then the following isomorphism holds.*

$$HF(\hat{M}, \hat{\mathcal{E}}) \otimes_{\mathbb{Z}_2} \Lambda^{\mathbb{Z}_2} \cong H(\mathscr{HOM}(\mathfrak{ID}, \Phi_{M_{12}})) \otimes_{\mathbb{Z}_2} \Lambda^{\mathbb{Z}_2}.$$

Here $\mathfrak{ID} : \mathscr{FUK}(R(\Sigma_{\text{in}}), R(\Sigma_{\text{in}}))$ is the identity functor and the right hand side is the homology of the chain complex consisting of A_∞ pre-natural transformations from $\mathfrak{ID}$ to Φ_{12}. (See [Fu6, Definition 7.49].*)*

Proof. (1) is obvious from the definition.

(2) We observe that $R(\Sigma) = R(-\Sigma)$ as spaces. On the other hand, the symplectic forms $\omega_{R(\Sigma)}$, $\omega_{R(-\Sigma)}$ and complex structures $J_{R(\Sigma)}$, $J_{R(-\Sigma)}$ are related by the formula:

$$\omega_{R(-\Sigma)} = -\omega_{R(\Sigma)}, \qquad J_{R(-\Sigma)} = -J_{R(\Sigma)}.$$

This implies (2) as follows. We remark that an (immersed) Lagrangian submanifold L of $(R(\Sigma), \omega_{R(\Sigma)})$ is also a Lagrangian submanifold L of $(R(\Sigma), -\omega_{R(\Sigma)})$. We consider the moduli space $\mathcal{M}_{k+1}((R(\Sigma), \omega_{R(\Sigma)}); L; \beta)$, which appeared (2.29) and is the case when we take $(R(\Sigma), \omega_{R(\Sigma)})$ as the ambient symplectic manifold.

We have an isomorphism

$$\mathcal{M}_{k+1}((R(\Sigma), \omega_{R(\Sigma)}); L; \beta) \cong \mathcal{M}_{k+1}((R(\Sigma), -\omega_{R(\Sigma)}); L; -\beta)$$

of spaces with Kuranishi structures. This isomorphism is defined by sending an element $(u; (z_0, z_1, \dots, z_k))$ of the left hand side to $(\overline{u}; (\overline{z}_0, \overline{z}_k, \overline{z}_{k-1}, \dots, \overline{z}_1))$. Here $\overline{u}(z) = u(\overline{z})$. Therefore evaluation maps

$$\text{ev} = (\text{ev}_0, \dots, \text{ev}_{k+1}) : \mathcal{M}_{k+1}((R(\Sigma), \omega_{R(\Sigma)}); L; \beta) \to L^{k+1}$$

satisfies $\text{ev}_{k-i} \circ I = \text{ev}_i$. (Our situation is somewhat similar to [FOOO6].) If we write the structure map of filtered A_∞ algebra of $L \subset (R(\Sigma), \omega_{R(\Sigma)})$ (resp. of $L \subset (R(\Sigma), -\omega_{R(\Sigma)}))$ by $\mathfrak{m}_k^+$ (resp. $\mathfrak{m}_k^-$) we have an equality

$$\mathfrak{m}_k^+(x_1, \dots, x_k) = \mathfrak{m}_k^-(x_k, \dots, x_1).$$

This implies that the filtered A_∞ algebra $(CF(L), \{\mathfrak{m}_k^+\})$ is an opposite algebra of $(CF(L), \{\mathfrak{m}_k^-\})$. We can generalize this fact to the case we have several Lagrangian submanifolds in a straight forward way. It implies (2).

(3) We remark that $\partial M_{12} = \Sigma_1 \sqcup -\Sigma_2$. Therefore $\partial M_{21} = -\Sigma_1 \sqcup \Sigma_2$.

The moduli space of the connection on the space in Figure 6.2, which we use to define $HF(\Phi_{M_{12}}(L_1, b_1), (L_2', b_2'))$ is isomorphic to the moduli space of the connection on the space in Figure 6.2, which we use to define $HF(\Phi_{M_{21}}(L_2', b_2'), (L_1, b_1))$. In fact such an isomorphism is obtained by the map which sends t to $-t$. (3) follows from this fact.

(4) follows from Theorem 1.1 and A_∞ Yoneda lemma [Fu6, Theorem 9.1] as follows. The diagonal $\Delta \subset R(\Sigma_{\text{in}}) \times R(\Sigma_{\text{out}})$ represents the identity functor. Therefore A_∞ Yoneda lemma implies

$$HF((\Delta, 0), (R(M_{12}, b_{12})) \otimes_{\mathbb{Z}_2} \Lambda^{\mathbb{Z}_2} \cong H(\mathscr{HOM}(\mathfrak{ID}, \Phi_{M_{12}})).$$

On the other hand, Theorem 1.1 (3) implies that the functor represented by $(\Delta, 0)$ is homotopy equivalent to the functor associated to $(\Sigma_{\text{in}} \times [0,1], b_{\Sigma_{\text{in}} \times [0,1]})$ by Theorem 1.3. Thus by Theorem 1.1 (2) we have

$$HF((\Delta, 0), (R(M_{12}, b_{12})) \cong HF(\hat{M}, \hat{\mathcal{E}}),$$

as required. $\square$

Theorems 6.2 and 6.3 show that the theory of relative $SO(3)$-Floer homology we developed in this paper satisfies (at least certain large portion of) the axioms of topological field theory.

We also remark that we can give an alternative proof of [BD, Theorem 2], (which is attributed to Floer) over $\mathbb{Z}_2$ coefficient from Theorem 6.3 (4) in the case Σ is a torus. In fact $R(T^2)$ consists of one point.

Remark 6.4. It seems that Wehrheim and Woodward proposed in [WW2] and several other papers, to use certain expected properties, which are similar to Theorems 6.2, 6.3 etc.. as an axiom and use it together with various results in differential topology to show the existence and uniqueness of a version of the relative Floer theory. It seems to the author that their idea is, in this way one may avoid studying gauge theory of 3 or 4 manifolds directly and can prove the results expected from the gauge theory by a combinatorial method.

An origin of such an idea is Floer's paper [Fl3], where Floer tried to use his Dehn surgery triangle as a main axiom to characterize Floer homology of 3-manifolds. (See [BD] for certain discussion about it.) The author in [Fu2], [Fu4] proposed to use this Dehn surgery approach to prove Theorem 1.1 (2), without using analysis so much. (This proposal by the author is not yet successful.)

The distinguished example where this kinds of idea works very much successfully is Heegard Floer theory by Ozvath-Szabo.

In this paper and in this subsection we take opposite route. Namely we define functors $\Phi_{M_{21}}$ directly by a geometric and analytic method and show its expected properties directly without using combinatorial method.

6.2. **Using similar moduli spaces.** In this paper, we use the moduli space introduced in [Fu5] or its variant to define and study Floer homology of 3-manifolds with boundary. There are several other moduli spaces which are similar to but is slightly different from that.

The moduli space studied by Lipyanskiy in [Ly] is somewhat of similar flavor. An element of his moduli space is also a combination of an ASD-connection and a pseudo-holomorphic curve. The difference is in place of the metric $\chi(s)^2 g_\Sigma + ds^2 + dt^2$ Lipyanskiy used direct product metric $g_\Sigma + ds^2 + dt^2$ and its ASD-connection. He instead introduce matching condition on the line where he switch from ASD-equation to pseudo-holomorphic curve equation. Lipyanskiy obtained also removable singularity and compactness results, which are mixture of Uhlenbeck and Gromov compactness. It seems likely that we can use Lipyanskiy's moduli space instead of one in [Fu5] to prove all the results of this paper, though the author did not check the detail.

In [Fu2] and [We1] the moduli space of different flavor is proposed and established, respectively. Namely we study the moduli space of ASD-connections on $M \times \mathbb{R}$, for example, where M has a boundary ∂M. We need certain boundary condition on $\partial M \times \mathbb{R} = \Sigma \times \mathbb{R}$. Such a boundary condition must be an 'infinite dimensional enhancement' of the Lagrangian submanifold of $R(\Sigma)$. The one which the author proposed in [Fu2] for this 'infinite dimensional enhancement' is different from one used in [We1]. With respect to this point, it seems that the one in [We1] (and not the one in [Fu2]) is the correct choice. The moduli space studied in [We1] can be used for problems related to those discussed in this paper.

However if we try to use it, in the same way as we are using the moduli space of [Fu5] in this paper, to prove the results of this paper, then there will be an issue. Let us explain this issue briefly.

Let L be an immersed Lagrangian submanifold of $R(\Sigma)$. We consider ASD equation on $M \times \mathbb{R}$. (Here the metric we use near $\partial M \times \mathbb{R}$ is $g_\Sigma + ds^2 + dt^2$ and does not degenerate.) We use L to define a boundary condition as in [We1]. Requiring asymptotic boundary condition as $t \to \pm\infty$ using $a_\pm \in R(M) \times_{R(\Sigma)} \tilde{L}$ in the same way as Definition 2.6 (3), we can define a moduli space $\mathcal{M}'((M,\mathcal{E}), L; a_-, a_+; E)$ and try to use it (instead of $\mathcal{M}((M,\mathcal{E}), L; a_-, a_+; E)$ in Definition 2.9) to define $HF((M,\mathcal{E}), L)$. As is shown by Salamon-Wehrheim [SaWe] this story works as far as L is embedded and monotone, since all the codimension one boundaries are of the form

$$\mathcal{M}'((M,\mathcal{E}), L; a_-, a; E_1) \times \mathcal{M}'((M,\mathcal{E}), L; a, a_+; E_2),$$

in that case. In case L is immersed or is not monotone, we need to combine this construction with the story of bounding cochains, since a disk bubble may produce codimension one boundary component. For this purpose it seems that we need to glue an element of $\mathcal{M}'((M,\mathcal{E}), L; a_-, a_+; E)$ with a pseudo-holomorphic disk which bounds L. Namely we need to introduce $\mathcal{M}'_k((M,\mathcal{E}), L; a_-, a_+; E)$, an analogue of $\mathcal{M}_k((M,\mathcal{E}), L; a_-, a_+; E)$ in Definition 2.16, (where k is the number of boundary marked points) and show that the fiber product

$$\mathcal{M}'_{k_1}((M,\mathcal{E}), L; a_-, a_+; E_1) \ {}_{\mathrm{ev}_i}\times_{\mathrm{ev}_0} \mathcal{M}_{k_2+1}(L; E_2) \tag{6.9}$$

appears at the boundary of $\mathcal{M}'_{k_1+k_2-1}((M,\mathcal{E}), L; a_-, a_+; E_1 + E_2)$.

The gluing analysis which we need to prove this statement is extremely difficult. Let me elaborate this point more. Let us start with an element $((\mathfrak{A}, \vec{z}), (u, \vec{z}'))$ of (6.9). The point $u(z'_0) \in R(\Sigma)$ is the gauge equivalence class of $\mathfrak{A}|_{\Sigma \times \{z_i\}}$. The usual method for gluing is to regard u as a family of flat connections and put it near $\Sigma \times \{z_i\}$ and try to glue it with $\mathfrak{A}$. The issue is we need to take a conformal diffeomorphism from a small neighborhood of z_i to the domain of u so that the family of flat connections u is supported in this small neighborhood of z_i after reparametrization. When we scale it so that the diameter of the domain of this family of flat connections becomes something like 1, then the metric becomes

$$\frac{1}{\epsilon^2} g_\Sigma + ds^2 + dt^2.$$

We observe that the family of flat connections u actually is very far from being a solution of the ASD-equation with respect to the scaled metric. It would be close to the ASD-connection if the factor $\frac{1}{\epsilon^2}$ were very small. However this factor is actually very large. By this reason the standard way of gluing does not seem to work.

We remark that the situation is different in the case when we consider the same gluing problem for the moduli space $\mathcal{M}_k((M,\mathcal{E}), L; a_-, a_+; E)$ in Definition 2.16, which we use in this paper. In fact, an element of this moduli space is $(\mathfrak{A}, \mathfrak{z}, \mathfrak{w}, \Omega, u, \vec{z})$ where u is a genuine pseudo-holomorphic curve to $R(\Sigma)$ in a neighborhood of the boundary. Therefore, gluing an element of $\mathcal{M}_k((M,\mathcal{E}), L; a_-, a_+; E)$ with a pseudo-holomorphic disk is similar to the gluing between two pseudo-holomorphic disks. In other words, if we have an appropriate Fredholm theory, we can work out the gluing analysis, in a way similar to those written in various literatures, eg. [FOOO2, Sections 7.1.4 and A.1.4]. The situation seems to be similar in the case of Lipyanskiy's moduli space.

On the other hand, if we restrict ourselves to the situation when $R(M)$ is embedded in $R(\Sigma)$ then it is likely that the moduli space of [We1] can used to prove for example Corollary 1.2. The proof could be in 4 steps.

(Step A) Let L_1, L_2 be two embedded monotone Lagrangian submanifolds in $R(\Sigma)$. Let $a_\pm \in L_1 \cap L_2$. We consider $\Sigma \times [-1,1] \times \mathbb{R}$ and study ASD connections on it under the boundary condition which is induced by L_1 on $\Sigma \times \{-1\} \times \mathbb{R}$ and L_2 on $\Sigma \times \{1\} \times \mathbb{R}$ and asymptotic boundary conditions given by a_- and a_+. Counting the solution in case its virtual dimension is 0, we obtain a matrix coefficient $\langle \partial^G a_-, a_+ \rangle$. It gives a boundary operator ∂^G on the $\mathbb{Z}_2$ vector space with basis $L_1 \cap L_2$. The monotonicity implies that $\partial^G \circ \partial^G = 0$. Let $HF(L_1, L_2)^{\text{gauge}}$ be its homology group. This step is already worked out in [SaWe], in a harder case when the bundle on Σ is a trivial $SU(2)$ bundle.

(Step B) We can use pseudo-holomorphic strip and define Floer homology of Lagrangian intersection $HF(L_1, L_2)$. (This step was done by Oh [Oh].)

(Step C) It may be possible to use adiabatic argument in a way similar to [DS] to show $HF(L_1, L_2) \cong HF(L_1, L_2)^{\text{gauge}}$.

(Step D) We consider the case $\partial(M_1, \mathcal{E}_1) = -\partial(M_2, \mathcal{E}_2) = (\Sigma, \mathcal{E}_\Sigma)$ and suppose $R(M_1)$ and $R(M_2)$ are both embedded Lagrangian submanifolds of $R(\Sigma)$. It seems that we can prove an isomorphism $HF(M, \mathcal{E}) \cong HF(R(M_1), R(M_2))^{\text{gauge}}$ in a way similar to Section 5 of this paper as follows. We consider the domain W as in Figure 5.1 and Condition 5.6. We glue $-M_2 \times \mathbb{R}$ to $\partial_2 W$ and $M_1 \times \mathbb{R}$ to $\partial_3 W$ in the same way to obtain the 4 manifold Y. This time we consider the direct product metric $g_\Sigma + ds^2 + dt^2$ on $\Sigma \times W$. We consider the set of gauge equivalence classes of connections, which are ASD connections on Y with respect to this metric and which satisfy the boundary conditions induced by $R(M_2)$ on $\partial_1 W$ and by $R(M_1)$ on $\partial_4 W$. We can use this moduli space to construct a chain map from $CF(M, \mathcal{E})$ to $CF(R(M_1), R(M_2))^{\text{gauge}}$, which is congruent to the identity map. Using the basic analytic results of [SaWe] it is very likely that we can work out the detail of this proof.

Let us consider the case when the bundle $\mathcal{E}_\Sigma$ is a trivial $SU(2)$ bundle, especially the case of handle body M. This is the case of Atiyah-Floer conjecture in its original form. In this case $R(\Sigma)$ is not a symplectic manifold since it has a singularity. Nevertheless as far as (Step D) concerns, there may not be so much big difference between this case and the case of nontrivial $SO(3)$ bundle on Σ. Namely we may be able prove the isomorphism

$$HF(M) \cong HF(R(M_1), R(M)_2))^{\text{gauge}} \tag{6.10}$$

as follows. Here M_1, M_2 are handle bodies whose boundaries are Σ. $R(M_i)$ is the space of flat connections on M_i regarded as a Lagrangian submanifold of $R(\Sigma)$. The closed 3 manifold M, which we assume to be a homology 3 sphere, is obtained by gluing M_1 and M_2 along Σ. The left hand side is a Floer homology of 3 manifold M defined by Floer [Fl1] and the right hand side is defined by Salamon-Wehrheim [SaWe]. Here we consider trivial $SU(2)$ bundles. We remark that (6.10) is [We2, Conjecture 4.3].

We consider $\Sigma \times W$ where W is as in Figure 5.1. We glue $M_2 \times \mathbb{R}$ and $M_1 \times \mathbb{R}$ to $\Sigma \times \partial_2 W$ and $\Sigma \times \partial_3 W$ respectively and obtain X. We put direct product metric $g_\Sigma + ds^2 + dt^2$ on $\Sigma \times W$ and extend it to Y.

For $a_-, a_+ \in R(M) = R(M_1) \cap R(M_2)$ we consider the moduli space $\mathcal{M}'(Y; a_-, a_+; E)$ of ASD-connections on Y of energy E such that:

(1) On $\Sigma \times \partial_1 W$ it satisfies the boundary condition of Wehrheim [We1] induced by $R(M_2) \subset R(\Sigma_1)$.

(2) On $\Sigma \times \partial_4 W$ it satisfies the boundary condition of Wehrheim [We1] induced by $R(M_1) \subset R(\Sigma_1)$.
(3) On the end where $\mathrm{Re}z \to +\infty$, it will converge to a_+.
(4) On the end where $\mathrm{Re}z \to -\infty$, it will converge to a_-.

Note at the end where $\mathrm{Im}z \to +\infty$ we use the fundamental class $R(M_2)$ as the asymptotic boundary condition and on the end where $\mathrm{Im}z \to -\infty$ we use the fundamental class $R(M_1)$ as the asymptotic boundary condition. It seems that these conditions are automatically satisfied from our assumption that the energy is finite. We however need some argument to prove it since $R(M_i)$ does not satisfy [SaWe, Condition L.3] when we consider M_i as the 3 manifold. (Note the 3 manifold which we denote by M_i above, is denoted by Y in [SaWe, page 748].)

Using the moduli space $\mathcal{M}'(Y; a_-, a_+; E)$ in case its virtual dimension is 0 we may obtain matrix elements of the map

$$\Phi : CF(M) \to CF(R(M_1), R(M_2))$$

between chain complexes defining left and right hand sides of (6.10). To show that this map Φ is a chain map we consider the moduli space, $\mathcal{M}'(Y; a_-, a_+; E)$, in case its virtual dimension is 1. We consider its boundary. The possibilities are

(I) ASD connections escape to the direction $\mathrm{Im}z \to +\infty$.
(II) ASD connections escape to the direction $\mathrm{Im}z \to -\infty$.
(III) ASD connections escape to the direction $\mathrm{Re}z \to -\infty$.
(IV) ASD connections escape to the direction $\mathrm{Re}z \to +\infty$.

Using monotonicity in the same way as the proof of Proposition 3.10 it seems that we can show that there is no contribution from the end of types (I) and (II). (We need some argument at this point also since $R(M_i)$ contains reducible connections.) (The monotonicity also implies that there is no contribution from the bubble at $\Sigma \times \partial_1 W$ and $\Sigma \times \partial_4 W$.)

The ends of type (III) (resp. of type (IV)) gives the boundary operator of the left hand side (resp. right hand side) of (6.10) composed with Φ. Therefore we may be able to prove that Φ is a chain map in this way.

The set of zero energy solutions $\mathcal{M}'(Y; a_-, a_+; 0)$ is the empty set if $a_- \neq a_+$ and consists of a single point a if $a = a_- = a_+$. Therefore Φ is congruent to the identity map modulo Λ_+.

We thus sketched a possible way to prove (6.10).

Note (Step A) in the case of handle bodies is established by Salamon-Wehrheim [SaWe].

Therefore, according to the author's opinion, (Step B) is the most difficult part which remains to be worked out to prove Atiyah-Floer conjecture based on the argument summarized above. Here (Step B) means: finding correct notion of Lagrangian intersection Floer homology for a pair of Lagrangian submanifolds in the singular space $R(\Sigma)$, by using pseudo-holomorphic map to $R(\Sigma)$. (Note this step had been done by Oh in case $R(\Sigma)$ is smooth.)

We also remark that in the proof of removable singularity theorem and compactness theorem in [Fu5], the author used the fact that all the elements of $R(\Sigma)$ are irreducible. So to use the moduli space of [Fu5] to study Atiyah-Floer conjecture in the original case of handle body, we first need to improve this point.

6.3. **Cobordism method and degeneration method.** In [DS] the key idea of the proof of their main theorem, which is equivalent to Corollary 1.2 in the case $M_1 = M_2 = \Sigma \times [0,1]$, is studying an adiabatic limit, where Σ collapses to a point. The method of this paper does not study this degeneration directly but replace it by a cobordism argument.[7] In other words, we bypass some of the difficult analytic issue by using cobordism argument. We remark that the idea proposed in [We2, Section 4] to prove Atiyah-Floer conjecture (in its original form) is based on degeneration analysis, which is harder than [DS].

A similar point appears also when we consider the related problem of the study of Wehrheim-Woodward functoriality ([WW1]). At the most important point of their work, Wehrheim-Woodward used strip shrinking, which is of a similar flavor as taking adiabatic limit. Then it appeared interesting and deep problem of figure eight bubble. In [WW1], figure eight bubble is excluded by using monotonicity. The main idea of Lekili-Lipyanskiy in [LL] is to replace difficult analytic problem of strip shrinking by a cobordism argument.

If we go beyond the monotone case, the effect of figure eight bubble becomes nonzero while studying strip shrinking. In the recent works by Bottman [Bo], Bottman-Wehrheim [BW], several important facts are discovered about figure eight bubble. There are certain heuristic discussions on the moduli space of figure eight bubbles in [BW].

The author conjectured that the moduli space of figure eight bubble gives a bounding cochain in the sense of [FOOO1] of the filtered A_∞ algebra associated by [AJ] to the immersed Lagrangian submanifold obtained by the Lagrangian correspondence.

If the author's conjecture is correct, then it is also very likely that this bounding cochain (up to gauge equivalence) coincides with one we obtained in Theorem 3.13. We emphasize that as far as analytic results concern we need only well-established or standard results to prove Theorem 3.13. Especially we do not need to study strip shrinking or figure eight bubble. In other words, we replace difficult analysis by an algebraic lemma[8] (Proposition 3.5). This is similar to Lekili-Lipyanskiy's argument which replaces strip shrinking by a cobordism argument. Actually the starting point of author's research, which leads to this paper, was to try to generalize Lekili-Lipyanskiy's method and combine it with the obstruction-deformation theory of [FOOO1] and immersed Lagrangian Floer theory of [AJ].

Acknowledgement. The research of the author is supported partially by JSPS Grant-in-Aid for Scientific Research No. 23224002 and NSF Grant No. 1406423.

References

[Ak] M. Akaho, *Intersection theory for Lagrangian immersions*, Math. Res. Lett. 12 (2005), 543–550.

[AJ] M. Akaho and D. Joyce, *Immersed Lagrangian Floer theory*, J. Differential Geometry 86 (2010) 381-500.

[At] M.F. Atiyah, *New invariants of three and four dimensional manifolds*, Proc. Symp. Pure Math. 48 (1988).

---

[7] However the proof of removable singularity and compactness theorem in [Fu5] uses a similar estimate as [DS].

[8] Studying figure eight bubble is certainly interesting for its own sake and can be applied in various places.

[BK] A. Bondal and M. Kapranov, *Framed triangulated categories*, Math. USSR. Sbornik 1 (1991 (1990 in Russian)), 93 - 107.
[Bo] N. Bottman, *Pseudoholomorphic quilts with figure eight singularity*, arXiv:1410.3834.
[BW] N. Bottman and K. Wehrheim, *Gromov compactness for squiggly strip shrinking in pseudoholomorphic quilts*, arXiv:1410.3834.
[BD] D. Braam and S. Donaldson, *Floer's work on instanton homology, knots, and surgery*, in Floer Memorial Volume, Hofer, Zender, and Taubes, Editors. 1995, Birkhäuser. pp. 195–256.
[D1] S. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geometry 26 (1987) 397–428.
[D2] S. Donaldson, *Connections, cohomology and the intersection forms of 4-manifolds*, J. Differential Geom. Volume 24, Number 3 (1986), 275-341.
[D3] S. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology 29 (1990), no. 3, 257–315.
[D4] S. Donaldson, *Gauge theory for non-athlete*, Lecture at University of Warwick, 1992.
[DS] S. Dostoglou and D.A. Salamon, *Self-dual instantons and holomorphic curves*, Annals of Mathematics 139 (1994), 581–640.
[Fl1] A. Floer, *Instanton invariant for 3-manifolds*, Comm. Math. Phys. 118 (1988), 215–240.
[Fl2] A. Floer, *Instanton homology and Dehn surgery*, in Floer Memorial Volume, Hofer, Zender, and Taubes, Editors. 1995, Birkhäuser. pp. 77–98.
[Fl3] A. Floer, *Instanton invariants for knots*, in Floer Memorial Volume, Hofer, Zender, and Taubes, Editors. 1995, Birkhüser. p. 99–104.
[Fu1] K. Fukaya, *Morse homotopy, A^∞-Category, and Floer homologies*, in "Proceedings of Garc Workshop on Geometry and Topology" ed. by H.J. Kim, Seoul National University (1994), 1-102.
[Fu2] K. Fukaya, *Floer homology for 3-manifolds with boundary -abstract -*, in 'Topology Geometry and Field theory' ed by Fukaya, Furuta, Khono, Kotchick, World Scientific, Singapore (1994), 1-22.
[Fu3] K. Fukaya, *Floer homology of connected sum of homology 3-spheres*, Topology 35 (1996) 88–136.
[Fu4] K. Fukaya, *Floer homology for 3-manifolds with boundary I* (1997), never to be published but is available at http://www.math.kyoto-u.ac.jp/%7Efukaya/fukaya.html.
[Fu5] K. Fukaya, *Anti-Self-Dual equation on 4-manifolds with degenerate metric*, Geometric Analysis and Functional Analysis 8 (1998) 466–528.
[Fu6] K. Fukaya, *Floer homology and mirror symmetry II*, Minimal Surfaces, Geometric Analysis and Symplectic Geometry (Baltimore, MD, 1999), Adv. Stud. Pure Math. 34 (2002), 31–127.
[Fu7] K. Fukaya, *$SO(3)$-Floer homology of 3 manifolds with boundary 2*, in preparation.
[Fu8] K. Fukaya, *$SO(3)$-Floer homology of 3 manifolds with boundary 3*, in preparation.
[Fu9] K. Fukaya, *Wehrheim-Woodward functoriality by Lekili-Lipyanskiy's method*, in preparation.
[FOOO1] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction, Part I,* AMS/IP Studies in Advaneced Math. vol. 46.1, International Press/ Amer. Math. Soc. (2009). MR2553465.
[FOOO2] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory-anomaly and obstruction, Part II,* AMS/IP Studies in Advanced Math. vol. 46.2, International Press/ Amer. Math. Soc. (2009). MR2548482.
[FOOO3] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Anchored Lagrangian submanifolds and their Floer theory.* in 'Mirror symmetry and tropical geometry', 15-54, Contemp. Math., 527, Amer. Math. Soc., Providence, RI, (2010).
[FOOO4] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds I*, Duke Math. J. 151 (2010), no. 1, 23-174.
[FOOO5] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory over integers: spherically positive symplectic manifolds*, Pure and Applied Mathematics Quaterly Volume 9, Number 2 189–289 (2013), arXiv:1105.5124.
[FOOO6] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Anti-symplectic involution and Floer cohomology*, submitted, arXiv:0912.2646.

[FOOO7] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Technical details on Kuranishi structure and virtual fundamental chain*, arXiv:1209.4410.

[FOOO8] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 1*, arXiv:1503.07631.

[FOOO9] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 2*, in preparation.

[FOn] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant*, Topology 38 (1999), no. 5, 933–1048.

[Gr] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), no. 2, 307–347.

[He] C. Herald, *Legendrian cobordism and Chern-Simons theory on 3-manifolds with boundary*, Comm. Anal. Geom., 2 (1994) 337–413.

[LL] Y. Lekili and M. Lipyanskiy, *Geometric composition in quilted Floer theory*, Adv. Math. 236 (2013), 1–23.

[LLW] R. Lee and W. Li, *Floer homologies for Lagrangian intersections and instantons*, preprint.

[Le] K. Lefèvre-Hasegawa, *Sur les A_∞ catégories*, Thése Doctorat Universite Paris 7 (2003).

[Ln] A. Lino, *The Künneth theorem for the Fukaya algebra of a product of Lagrangians*, arXiv:1407.8436.

[LOT] R. Lipshitz, P.S. Ozsvath and D. Thurston, *A tour of bordered Floer theory*, Proc. Nat. Acad. Sci. 108 (2011) 8085–8092.

[Ly] M. Lipyanskiy, *Gromov-Uhlenbeck Compactness*, arXiv:1409.1129.

[MMR] J. Morgan, T. Mrowka and D. Ruberman, *The L^2-moduli space and a vanishing theorem for Donaldson polynomial invariants*, International Press, 1994.

[MWW] S. Ma'u, K. Wehrheim and C. Woodwards, *A_∞-functor for Lagrangian correspondences*, preprint.

[Oh] Y.-G. Oh, *Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I, II*, Comm. Pure and Appl. Math. 46 (1993), 949–994, 995–1012; Addenda, ibid, 48 (1995), 1299 –1302.

[Sa] D.A. Salamon, *Lagrangian intersections, 3-manifolds with boundary, and the Atiyah-Floer conjecture*, Proceedings of the ICM, Zürich 1994, Vol. 1, 526–536.

[SaWe] D. Salamon and K. Wehrheim *Instanton Floer homology with Lagrangian boundary conditions*, Geom. Topol. 12 (2008), no. 2, 747–918.

[Uh1] K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. 83 (1982), no. 1, 31–42.

[Uh2] K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. 83 (1982), no. 1, 11–29.

[We1] K. Wehrheim, *Anti-self-dual instantons with Lagrangian boundary conditions I: Elliptic theory II : Bubbling*, Comm. Math. Phys. 254 (2005), no. 1, 45–89, ibid. 258 (2005), no. 2, 275–315.

[We2] K. Wehrheim, *Lagrangian boundary conditions for anti-self-dual instantons and the Atiyah-Floer conjecture*, Conference on Symplectic Topology. J. Symplectic Geom. 3 (2005), no. 4, 703–747.

[WW1] K. Wehrheim and C. Woodward, *Quilted Floer cohomology*, Geometry & Topology, 14:833–902, 2010.

[WW2] K. Wehrheim and C. Woodward, *Floer field theory for tangles*, arXiv:1503.07615.

[Ye] R. Ye, *Gromov's compactness theorem for pseudo holomorphic curves*, Trans. Amer. Math. Soc. 342 (1994), 671–694.

[Yo] T. Yoshida, *Floer homology and holomorphic curves - Atiyah conjecture*, preprint, 1992.

Simons Center for Geometry and Physics, State University of New York, Stony Brook, NY 11794-3636 U.S.A. & Center for Geometry and Physics, Institute for Basic Sciences (IBS), Pohang, Korea

E-mail address: kfukaya@scgp.stonybrook.edu